Impacts of students’ difficulties in constructing geometric concepts on their proof’s understanding and proving processes

(article begins on next page)

PROCEEDINGS OF THE 40TH CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

EDITORS: CSABA CSÍKOS • ATTILA RAUSCH • JUDIT SZITÁNYI

VOLUME 2

PME40 / SZEGED / HUNGARY
3-7 AUGUST, 2016

Proceedings of the 40th Conference of the

International Group for the Psychology of Mathematics Education

PME40, Szeged, Hungary, 3–7 August, 2016

Mathematics Education:
How to solve it?

Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education

Editors

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PME40, Szeged, Hungary, 3–7 August, 2016

Cite as:

Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2. Szeged, Hungary: PME.

Website: http://pme40.hu

The proceedings are also available via http://www.igpme.org

Publisher:

International Group for the Psychology of Mathematics Education

ISSN 0771-100X
ISBN 978-1-365-45852-1

Logo: Lóránt Ragó
Composition of Proceedings: Edit Börcsökné Soós

Printed in Hungary
Innovariant Nyomdaipari Kft., Algyő
www.innovariant.hu

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Volume 2
RESEARCH REPORTS
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MATHEMATICS PROBLEM POSING SKILLS IN SUPPORTING PROBLEM SOLVING SKILLS OF PROSPECTIVE TEACHERS

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The activity of posing and solving problems can enrich teachers' mathematical experiences because it fosters a spirit of inquisitiveness, cultivates their mathematical curiosity, and deepens their views of what it means to do mathematics with learners. Contemporary reform efforts not only place a heavy emphasis on problem solving but also on problem posing. The suggestions in both standards (NCTM, 1991, 1995) imply that problem posing in an integral part of problem solving should not be emphasized separately from problem solving.. The purpose of this study was to examine the effectiveness of carrying out problem posing skills on prospective mathematics teachers problem solving performance and, especially to find out whether there were differences between those who used problem posing strategies and those who did not. The results of this study showed that the performance of problem solving for prospective mathematics teachers improved overall when using their problem posing skills.

INTRODUCTION

One of the major goals in mathematics teaching is to encourage our students to be good problem solvers. To achieve that goal teachers have to teach mathematical problem solving strategies with more practice. Mathematics educators tend to neglect the other side of the coin in mathematical problem solving in mathematics teaching programs, that is problem posing (Gonzales, 1994), in spite of its importance in developing our students' mathematical thinking. New trends in mathematics education (NCTM, 2000) recommend a change from asking students to solve problems, to developing problem through changing their questions, adding new data, eliminating some data, changing variables or construction a new problem based on the original idea.

In the author's discussions with teachers, the author observed that their abilities in solving non-routine problems were very weak. But they had a positive attitude to pose questions from a given problem. The author tried to give more attention in mathematical problem solving when posing a topic in "Methods of Teaching mathematics" for prospective teachers in the College of Education.

OBJECTIVES OF THE STUDY

1) To identify the effectiveness of using problem-posing skills on performance of prospective mathematics teacher for problem solving.
2) To identify problem posing skills needed to be included with Polya's four steps to improve mathematics for prospective teachers in problem solving performance.
3) To develop educational activities for mathematical problem solving and posing as a part of mathematics education program for prospective teachers.

BACKGROUND

The first recommendation in " An Agenda for Action" produced by the NCTM in the US, recommended problem solving be the focus of school mathematics in the 1980s. School Mathematics should contain problem solving as the main activities in all mathematics aspects; also teachers should offer their students rich problems, often based in the real world, which

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 3–10. Szeged, Hungary: PME.

would challenge and excite them, because problem solving is an effective way to introduce and explore new areas in mathematics. Through problem solving, the students can develop much of the mathematics for themselves.

Student teachers are prepared to teach mathematics with a problem solving approach, to help their students in solving mathematical problems. Their educational program to do that doesn't reflect their abilities to solve problems. Abilities to use different problem posing strategies, may affect their problem solving performance.

Relationships between problem solving performance and problem posing still need to be explored as Silver and Cai (1993) mentioned " there is a need for further research that examines the complex relationship between problem posing and problem solving." There is also interest in exploring the relationship of posing to other aspects of mathematical knowing and mathematical performance.

In silver's (1994) researches, he found different results of that relationship. Silver and Cai (1993) found a strong positive relationship between posing and solving performance. While Silver and Mamona (1989) found no overt link between the problem posing of middle school mathematics teachers and their problem solving abilities there is no clear, simple link established between competence in posing and solving problems(Silver, 1994). It is possible to improve student teachers' performance in problem solving, by using problem posing strategies. Kilpatrick discussed that and suggested that by drawing students' attention to the reformulating process and given practices in it, the students can improve problem solving performance (1987).

Given a mathematical problem to a student, means the student is put in a new thinking situation; thinking of the given information in the problem statement, thinking of a best strategy to solve it using his own questions that lead him to a solution and thinking of more information related to the given information.

The given information given explicitly in a problem statement is almost never adequate for solving the problem. The problem solver has to supply additional information consisting of premises about the problem context (Kilpatrick, 1987). For example, to solve a word problem about the distance between two cities, students need to understand that distance cannot be negative numbers.

The idea of improving mathematical problem solving performance has been discussed in the light of Polya's four steps for problem solving. Through problem posing in Polya's steps, problems can themselves be the source of new problems. The solver can intentionally change some or all of the problem condition to see what new problem results, and after a problem has been solved the solver can "look back" to see how the solution might be affected by various modification in the problem.

In "making a plan" to solve a problem, Kilpatrick (1987) showed that students may take Polya's heuristic to see whether, by modifying the condtion in the problem, a new, more accessible problem might result that could be used as a stepping stone to solve the original one.

Polya was looking towards problem solving as a major theme of doing mathematics, and "teaching students to think" was of primary importance. The other aspect of problem solving that is seldom included in textbooks is problem posing. Polya did not write specifically on problem posing, but much of the spirit and format of problem posing is included in his illustrations of "looking back" (Wilson, Fernandez & Hadaway, 1993). "Looking back" may be the most important part of problem solving. It is the set of activities that provides the primary opportunity for students to learn from the problem. Polya identified this phase with admonitions to examine the solution by such activities as checking the result, checking the argument, deriving the result differently, using the method for some other problem, reinterpreting the problem or stating a new problem to be solved.

Teacher's skills on using Polya's four steps in problem solving should be consistent with their abilities to use suitable problem posing strategies to generate more questions and problem for students.

MATHEMATICAL PROBLEM POSINGS STRATEGIES

Mathematics teachers might use one or more strategies to formulate new problems or encourage their students in mathematics classes to be good problem posers as well as good problem solvers. Strategies could be used depending on the most suitable conditions (mathematics content, students, levels, learning outcomes and mathematical thinking, types). Problem posing situations are classified as free, semi-structured or structured situations.

Free Problem Posing Situations

Situations from daily life (in or outside school) can help a student to generate some questions leading him/her to construct a problem. Students are asked to pose a problem to encourage them to "make up a simple or difficult problem" or " construct a problem suitable for a mathematics competition (or a test)" or " make up a problem you like." It is more useful if the teacher tries to relate the real life situations to the mathematics content being taught and to ask student to pose new problems. This will be more effective in developing students' mathematical thinking. Problem posing situations might take these types: everyday life situation, free problem posing, problem they like, problems for a mathematics competition, problems written for a friend and problems generated for fun.

Semi-Structured Problem Posing Situations

Students are given on open-ended situation and are invited to explore it using knowledge, skills, concepts and relationships from their previous mathematical experiences and this can take the following forms: Open-ended problems (i.e. mathematical investigation). Problems similar to given problems. Problem with similar situations. Problems related to specific theorems. problems derived from given pictures.word problems. this strategy was developed with prospective teachers as the following (Abu-Elwan,1999):

1) A semi-structured situation from a student's daily life was presented to all students.
2) Students were asked to complete the situations using their perspective to be able to pose problem from that formed situation.

Students can generate problems by omitting the questions from given situations.

Structured Problem Posing Situation

Any mathematical problem consists of known (given) and unknown (required) data. The teacher can simply change the known and pose a new problem, or keep the data and change the required. Brown and Walter (1990, 1993) designed an instructional problem formulating approach based on the posing of new problems from already solved problem, but they have also recommended varying the conditions or goals of given problem. This reformulation approach appears to be the most effective method for introducing structured problem posing activities in mathematic classrooms.

In order to create teaching/learning situations that provide a good problem posing situations, Lowrie (1999) recommended the mathematics teacher to:

1) encourage students to pose problems for friends who are at or near their own level until they become more competent in generating problems;
2) ensure that students work cooperatively in solving the problems so that the problem generator gains feedback on the appropriateness of the problem they have designed;
3) ask individuals to indicate the type of understanding and strategies the problem solver will need to use in order to solve the problem successfully before a friend generate a solution;
4) encourage problem solving teams to discuss, with one another, the extent to which they found problems to be difficult, confusing, motivating or challenging;
5) provide opportunities for less able students to work cooperatively with a peer who challenged the individual to engage in mathematics at a higher level than they were accustomed;
6) challenge students to move beyond traditional word problems by designing problems that are open ended and associated with real life experiences ; and
7) encourage students to use technology (calculators, CDs, computers) in developing their mathematical thinking skills, so they can use this technology to generate new mathematical situations.

RESEARCH QUESTIONS

This study attempts to answer the following questions:

1) How effective is the teaching of problem posing strategies able to enhance student teachers' performance in problem solving?
2) Is there any differences in mathematical problem solving skills between student teachers who study problem posing strategies and those who just study problem solving strategies?

Hypotheses

The study included three hypotheses:

1) There is a statistically significant difference ($p<0.01$) between student-teachers' mean scores of the experimental group and the control group in the mathematical problem solving part of the achievement text in favor of the experimental group.
2) There is a statistically significant differences ($p<0.01$) between student-teachers' mean scores of the experimental group and the control group in mathematical problem posing part of the achievement test in favor of the experimental group.
3) There is a statistically significant differences ($p<0.01$) between student-teachers' mean scores of the experimental group and the control group in mathematical problem (solving-posing) test in favor of the experimental group.

METHOD

Subjects

Fifty prospective teachers participated in the study. All of them were in Grade Three in the College of Education, (Sultan Qaboos University) and their major subjects were mathematics and computer science. They were enrolled in the "Methods of Teaching Mathematics II" Course. They were divided in two groups. Group E as an experimental group and Group C as a control group: each group consists of 25 student teachers.

Instruments

An achievement test on "Mathematical Problem Posing-Solving" had been developed to determine the uses of problem posing skills to enhance student teachers' performance in mathematical problem solving. The main topics and ideas of the achievement test are relevant in the prospective teachers' everyday life and are reinforced through common activities like providing shopping and vacationing problems developed based on the three problem solving strategies: Look for a pattern, Make a list, Work backward. The test consists of eight open-ended problems, each one of these problems contains: A statement of the problem, A question asking the students to solve it, Another question asking students to pose an extension to the original problem, then to solve it.

Four referees were asked to give their opinions regarding the validity of the test. Based on their suggestions, all modification were developed. The reliability of the test was established using a group of 20 mathematics teachers. A reliability coefficient mean value of 0.69 was secured. The achievement test in its final form was then used in the experimental design.

The Experimental Design

First: Student-teachers in Group E and C were involved in a quasi-experimental design as follows:

Student-teachers in Group C (control) studied a problem solving strategy based on Polya's four steps; several problems has been presented to student-teachers in this group, they were requested to solve the problems using suitable problem solving strategies. The problems had been chosen from:

- School mathematics textbooks (middle and secondary stages)
- Problem solving experiences in mathematics as a source book
- The internet web: www3.actden.com/math-den

Second**:** student-teachers in group E (experimental) studied problem solving strategies based on Polya's four steps (polya' 1973). Problems had been chosen from the previous sources like group C. But students in group E, studied it under other treatment for Polya's four-steps. Student-teachers explored the intent of each step in the process and attained a basic understanding of the process (Figure 1).

The students are introduced to solve the problems by traversing each of Polya's steps in the problem solving process.

They are required to write a description of each step as follows.

1) Understanding the problem:
   Ask yourself question such as: "what is the problem all about?"
   "What am I given and not given?" "What do I need to find?"
2) Make a plan:
   What strategy of which you know (look for pattern, making table or work backward) you will use.
3) Carrying out the plan:
   Perform the necessary computations and describe the steps that you take.
4) Evaluate solutions:
   Check if there might be other solutions or other strategies which will yield the same solution.

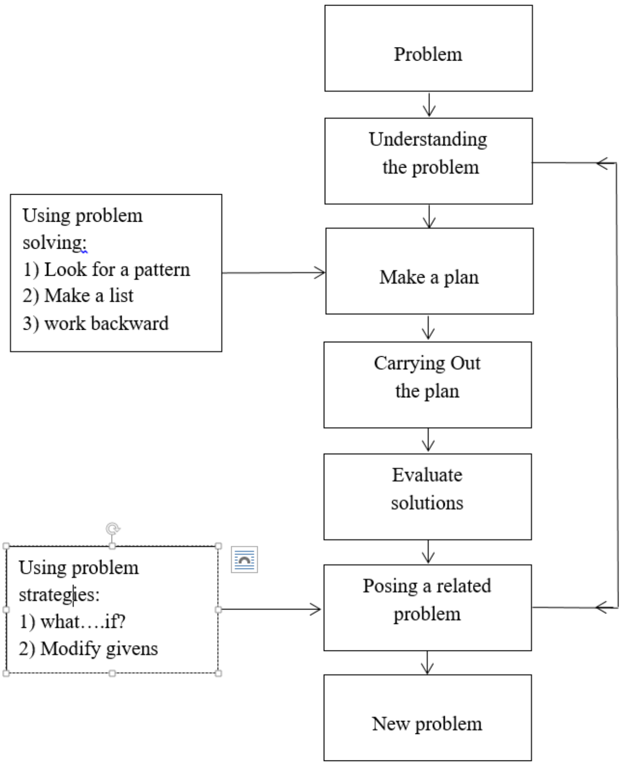

Figure 1: Framework for cycle of activities (solving-posing)

Student teachers must indicate all questions, attempts, frustration or any restriction they may have placed on a problem. Within the context of solving a given set of problems, probing questions are posed such as: Are all the given data relevant to the solution? Do any assumptions have to be made? Are there different ways interpreting the given information or conditions? As the questions are posed, students reach a good understanding of each problem.

The most important step is to encourage students to "generate an extension of the given problem" or "posing a related problem" as is suggested by Gonzales (1994). She suggested a fifth step which is:

5) Posing a related problem:

   Use the given problem and modify it to obtain a variation of the given problem. A student poses a related problem by change the values of the given data and by changing the context of the original problem. That doesn't mean he has to modify or change the solving strategy used in the original problem.

Student teachers may use any of the following techniques in writing new related problem: Change the values of the given data, Change the context, and Change the number of conditions that relate to the problem.

***Third*:** The Group C students have studied "problem solving activities" during the period September to October 2000 (5-weeks) while Group E had studied "problem solving-posing activities" during the period September to October 2000 (7-weeks), the extra two weeks was because of the techniques used in problem posing during and after solving the problem given.

***Fourth*:** An achievement test on "Mathematical problem solving-posing test" was presented to student teachers in Groups C and E in the same time as a post-test.

Results and Discussion:

To determine the effectiveness of problem posing strategies on prospective mathematics teachers' problem solving performance, the t-test was used as a measure of comparison between the mean scores on the mathematical problem (solving-posing) test for both the experimental group E and the control group (C).

The SPSS V19.0 had been used and the results are given in Table1.

Variable	Experimental (n=25)		Control (n=25)		Mean Difference	t-value	p<
	M	SD	M	SD			
Problem solving	4.16	.75	3.64	1	.52	2.09	0.01
Problem posing	2.32	.69	1.76	.66	.56	2.93	0.01
Problem (solving-post)	8.16	1.43	6.72	1.62	1.44	3.33	0.01

Table 1 Means, Standard Deviations and the Value of t Scores in Mathematical problem (Solving-Posing) Test.

Maximum score in "problem-solving part" of the test= 5 marks
Maximum score in "problem-posing part" of the test = 3 marks
Maximum score in "problem (solving-posing) test" = 10 marks

It is evident from Table 1 that the level of problem solving performance has significantly improved for student teachers of the experimental group compared do the performance of other student teachers of the control group in problem solving, with means of 4.16 and 3.64 respectively. The resulting t value of 2.09 is significant at $p<0.01$. This improvement in the level of problem solving performance for the experimental group is consistent with the significant increase in problem posing performance for student teacher in the control group, with means of 2.32 and 1.76 respectively. The resulting t value of 2.93 is significant at $p<0.01$. Student teachers of the experimental group were able to exhibit a significantly higher level or problem-solving performance as compared to that exhibited with student-teachers of control group.

Consistent with the previous result, student teachers of the experimental group were able to exhibit a significantly higher level of problem-posing performance as compared to that exhibited with student-teachers of control group.

Overall, the level of problem (solving-posing) performance has significantly improved for student teachers of the experimental group comparing with performance of student teachers of control group in the test of problem (solving-posing) with means of 8.16 and 6.72 respectively.

The resulting t value of 3.33 is significant at $p<0.01$. This study aimed at examining the effectiveness of problem posing strategies on prospective mathematics teachers problem solving performance.

The study sought to develop Polya's four-steps method to include more questions in "make a plan" and "carrying out the plan" or make an extension to the original problem to enhance student teachers performance in problem solving. Participants of experimental group have studied problem solving using Polya's four-steps as it is developed by problem posing strategies.

Results from this study supported the hypothesis and a significant improvement in student teachers problem solving performance was observed, as well as there is a significant improvement in student teachers problem posing performance in the achievement test.

An alternative explanation for the present finding is that student-teachers have opportunities to discuss each step in their solving a presented problem with more emphasis on the uses of problem posing strategies used to develop new problem, that support the same findings (Gonzales, 1994; Leung, 1993).

More researches need to be performed on the relationship between problem solving performance and problem posing abilities for students in all stages.

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STUDY HABITS IN UNDERGRADUATE MATHEMATICS: A SOCIAL NETWORK ANALYSIS

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This paper presents an exploratory social network analysis of the study behaviours of undergraduate mathematics students. Focusing on the second-year students within a large lecture class, it presents data on their self-reported percentage lecture attendance, number of hours spent studying alone and with others outside lecture time, and occasional and frequent conversations about mathematics with other students. It then presents analytical results on relationships between individuals' centrality within the network and amount of study time.

INTRODUCTION

Many factors affect students' engagement with undergraduate mathematics. Some of these factors are cognitive: we might expect students with better prior performance or logical reasoning skills to perform better in assessments (Alcock, Bailey, Inglis & Docherty, 2014). Some factors involve individual behavioural characteristics: we might expect students who are more conscientious (Alcock, Attridge, Kenny & Inglis, 2014) or who have better study skills (Credé & Kuncel, 2008) to engage more consistently and effectively with learning resources and mathematical ideas.

Some important factors, however, are not individual but social. Undergraduates do not study in isolation: their peers form an important part of the learning environment, and students are likely to share knowledge and influence one another's study habits. Mathematics educators are concerned about the role of such interactions in learning, as evidenced by research on mathematical discourse (Ryve, 2011) and by commentaries about the 'social turn' in mathematics education (Lerman, 2000) and the socio-political environment in which learning occurs (Gutiérrez, 2013). But published work in this area has been primarily theoretical (e.g. Cobb, Stephan, McClain & Gravemeijer, 2011; Morgan, 2006); where empirical studies are reported, these often involve detailed qualitative reports of short sequences of interactions in single classrooms (Ryve, 2011). Social interactions are not usually documented on a large scale or used quantitatively to predict educational outcomes.

But it is perfectly possible to study questions about social interactions at a larger scale, and *social network analysis* (SNA) provides one way to do this (Borgatti, Everett & Johnson, 2013; Carolan, 2013). The basic approach is simple: participants are asked to state with whom they interact around one or more issues, and (perhaps) how often or with what intensity they interact with those people. They might also be asked straightforward questions about their individual habits or characteristics. This information is then used to build a social network model and to investigate questions at the whole cohort level (describing features such as the density of the network or

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 11–18. Szeged, Hungary: PME.

changes over time), at the dyad level (using individual characteristics to predict who interacts with whom), or at the individual level (using individual characteristics plus amount or quality of interaction to predict an outcome measure such as performance). In the study reported here, we used SNA to conduct an exploratory investigation of the interactions between students in a large undergraduate mathematics class. This paper reports early and primarily descriptive findings; at the conference we also expect to report in more detail on a broader range of analytical questions.

THEORETICAL BACKGROUND

SNA studies typically use one of two basic approaches: egocentric studies use representative sampling to select people who will report on their individual social networks; whole-cohort studies use data from every person – as far as is practical – in some well-defined group in order to construct a social network model for that group (Carolan, 2013). Studies of both types have been conducted in education. For instance, Coburn, Choi and Mata (2010) collected data from elementary school teachers involved in a mathematics reform effort, and documented a shift over time from small, homogenous, grade-level-focused networks to larger, more diverse networks influenced by a desire to interact with those with expertise; Thomas (2000) collected data on social interactions between first-year students at a liberal arts college, and found that those who were better connected were less likely to drop out.

The SNA approach is arguably of particular relevance in higher education, because the greater requirement for independent study renders student-student interactions more important than they might be for younger pupils. For instance, at the UK university in which the present study took place, mathematics students spend approximately 18 hours per week in lectures and tutorials, and lecture classes can involve over 200 students. This means that around half of the students' learning is supposed to occur during independent study time, and that most students have little individual contact with lecturers. Students' peers therefore become a natural resource for information about both mathematical ideas and practical matters; students could influence one another substantially, especially if they study together regularly.

At present we know very little about student-student interactions outside lectures, or about how mathematics students use their independent study time more broadly. Research in undergraduate mathematics education has typically focused on instructional inputs: on what the teacher does in the classroom (Ellis, Kelton & Rasmussen, 2014; Johnson, Caughman, Fredericks & Gibson, 2013), or on the design of tasks to engage students in sophisticated mathematical reasoning (Larsen, 2013; Zandieh & Rasmussen, 2010), or on students' uptake of provided learning resources (Inglis, Palipana, Trenholm & Ward, 2011). Some of this work has an important social dimension – task designers often want students to work together to reinvent mathematical ideas. But attainment in the broader sense will also likely depend upon the nature and effectiveness of independent study conducted outside the classroom.

The work reported here forms part of a wider study addressing questions about which students study together and about whether being better connected is associated with better academic performance. This paper reports early results on three questions:

1. What proportions of lectures do students attend, how much time do they dedicate to independent study, and how much of this time do they spend collaborating with other students?
2. How interconnected is the mathematical interaction network, and what is the distribution of connectedness of individual students?
3. Do better-connected students study for more hours?

METHOD

Data for this study was collected in a real analysis lecture course that was compulsory for all students in the first year of a mathematics programme and either compulsory or optional for many students in the second years of programmes combining mathematics with other subjects. The lecturer displayed partially-populated notes on a visualiser, students had paper copies, and lectures mixed traditional lecturing with short activities for students to complete in collaboration with their neighbours. All students attended the same lectures (there were no "sections"), and all were exposed to different lecturing styles in other courses. Lecture attendance was not monitored; typical attendance in the Analysis course was 160-170 out of 214 registered students.

To maximise the number of students present to take the survey – an important consideration for whole-cohort SNA studies (Borgatti, Everett & Johnson, 2013) – two steps were taken. First, data collection took place in week 9 of the 11-week term, on the day of one of three small, summative in-class tests (students who are often absent do normally attend on test days). Second, three weeks earlier, on the day of the previous test, the researchers were introduced, the plan for the study was explained, and students were shown the form they would be asked to complete and offered the opportunity to ask questions about the project.

Participants were asked to complete a two-part survey. Part 1 asked them to report their approximate percentage lecture attendance, approximate number of hours per week spent on independent study, and approximate number of those hours spent working with other students. Part 2 asked them to list other students in the class with whom they spoke about mathematics frequently (once per week or more) and occasionally (less than once per week). Asking participants about one another in this way raises ethical issues that do not arise for surveys in which participants report only on their own habits (Carolan, 2013). Thus the informed consent form attached to the survey made clear that if a student did not sign or return the form, any data provided about them by others would not be used[1].

[1] Also, the only person to handle the original data was a lecturer not involved in teaching these students; the remaining team members worked only with the anonymised data.

To produce a relatively small dataset for this short paper, we restrict our attention to the second-year students; 77 of the 214 registered students were in this category, and 45 signed and returned their forms. These students had shared numerous lectures and tutorials in the previous year, so they form a somewhat coherent subset of the whole cohort, and they are more likely to have stable pre-existing study relationships. Data presented below come from both parts of the survey. Data from the social network part provided a total of 126 listed connections from one second year student to another[2]. These links required further processing because people have imperfect memories so it can easily be the case that one person reports an interaction that another has forgotten, or that one person reports frequent interaction where another reports only occasional interaction. In this case the researcher has a choice: work with directed links (allowing different weights to be attached to the link from A to B and that from B to A) or to assume that interactions are more likely to be forgotten than invented and to simplify by symmetrising, taking the highest level of reported link to be the accurate report (Carolan, 2013). In this report we take the second, simplifying approach.

RESULTS

In line with our research questions, we present the results in three sections: attendance and independent study time, network features and individual connectedness, and relationships between connectedness and study time.

Attendance and independent study time

Self-reported lecture attendance and independent study time varied widely. Reported lecture attendance ranged from 20% to 100%, with a median of 85% and a mean of 81%. Reported weekly hours of independent study ranged from 2 hours to 44 hours with a median of 10 hours and a mean of 13 hours. Because all students had roughly 18 hours of scheduled lecture and tutorial time, this means that the typical student claimed to study for approximately 25-28 hours per week. It is worth noting that this is less than would typically be recommended by their instructors.

Within independent study time, distribution of work alone versus work with others also varied widely. The graph in Figure 1 shows reported weekly hours studying collaboratively with others against reported total weekly hours of independent study. Naturally, students who study for more hours in total have more capacity to spend time studying with others, but there was no clear relationship beyond that: percentage of independent study time spent in collaborative study was not systematically related to total study time (r = -.010, p = .516).

[2] Of the 49 second-year participants, 3 reported no links, 7 reported a total of 12 links to other second-year students who did not sign the consent forms (and thus were not used) and 1 listed a link to a first-year student (which is not reported here but will be considered in future analyses).

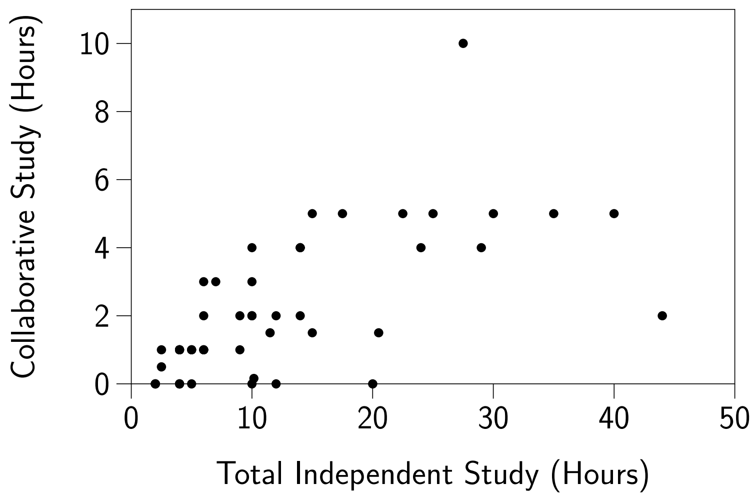

Figure 1: Hours studying collaboratively against total independent study hours.

Network features and individual connectedness

The network data was used to produce the graph in Figure 2; each node represents a student, thick lines represent reported frequent interactions and thin lines represent reported occasional interactions; isolates within this dataset are shown in the top left.

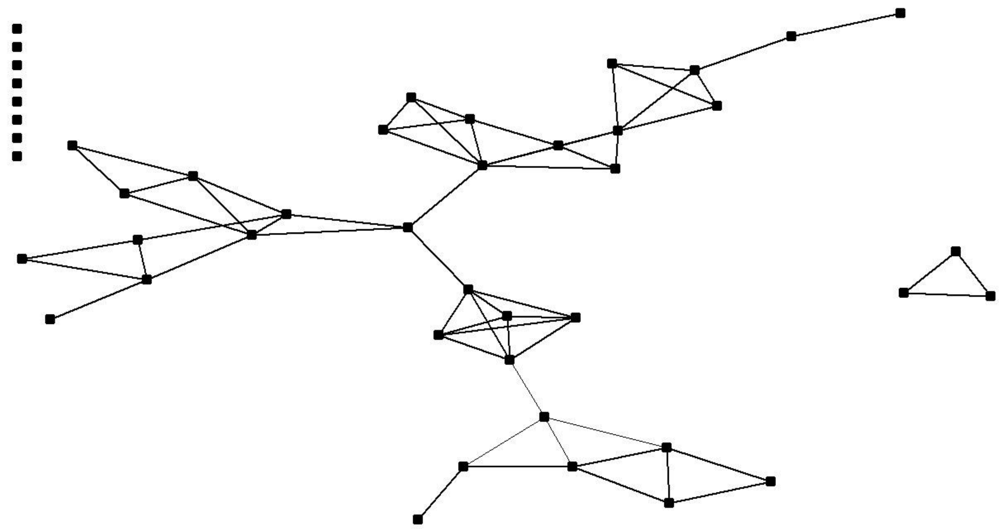

Figure 2: Students (nodes), frequent mathematical interactions (thick lines) and occasional interactions (thin lines).

One feature of this graph is that most students who participated are connected to one another, if only distantly. Another is that the nature of the connections varies widely. Some students are central in highly interconnected parts of the network, some are peripheral to these denser areas, and some study either alone or in a much smaller group[3]. This phenomenon can be captured quantitatively using various measures of

[3] We remind the reader that these are reported links from participants only: although they provide a strong sense of variability in the network, unreported links could give a different picture and, if the paper is accepted, we will discuss this potential problem at the conference.

centrality (Carolan, 2013), the simplest of which is *degree* in the graph theory sense: the degree of a node or vertex is the number of links it has to other nodes. Table 1 displays the distribution of degrees across the 45 participants.

Table 1: The number of participants with each degree (number of connections).

Degree	0	1	2	3	4	5	6	7
Number of Participants	8	3	7	9	13	3	1	1

Connectedness and study time

The remaining question we address in this paper is whether better connected students study for more hours. It is plausible that they would: perhaps being part of regular mathematical conversations supports engagement with study. It is also plausible that they would not: people who tend to talk more might be more generally sociable and spend more time on alternative activities. Our data showed that reporting more connections is associated with more hours of collaborative work: degree is correlated with hours spent in collaborative independent study (r = .450, p = .002), meaning that those who spend more time talking about mathematics with others also study with more different people. But degree was not, in our data, linked to doing more or less independent study overall (r = .053, p = .731). The graph in Figure 3 illustrates this.

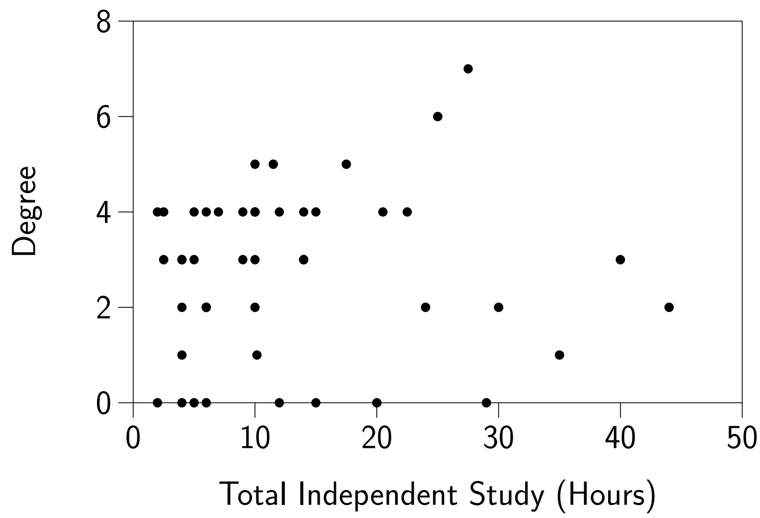

Figure 3: Degree against total independent study hours.

DISCUSSION

This paper has documented undergraduate mathematics students' self-reported independent study time and collaboration with others. This work provides a first picture of the participants' habits, but also raises many questions about the reasons for students' choices and their effects on subsequent academic attainment. We are exploring these issues and expect to report on the larger dataset at the conference.

For now, we believe that research of this nature has potential implications for study guidance, because – despite the existence of study guidebooks (e.g. Alcock, 2013) – there is little published evidence on how mathematics students can profitably spend their independent study time. Is it better to study with other students or alone? Is it a good idea to converse frequently with a small number of others, because communication improves as people become more familiar with one another's thinking? Or is it better to have occasional conversations with many, because this provides access to a broader range of knowledge and understanding? Further, if research makes it possible to provide guidance about what works best on average, do exceptions exist? Can any student attain high levels of success by studying entirely in isolation, or is this likely only for students with particular individual characteristics or particular levels of prior attainment? A broad base of research addressing such questions would make it possible to structure student activities so as to support productive interactions, and to provide students with advice that is both evidenced and nuanced.

We also believe that research of this nature has potential for developing knowledge in mathematics education. The field has lately evolved to a state that is effectively dichotomised: researchers interested in social issues typically use qualitative methods (see e.g. Ryve, 2011); quantitative studies are routinely conducted by those with backgrounds in psychology but are comparatively rare in mathematics education journals (Alcock, Gilmore & Inglis, 2013). This no doubt occurs partly for reasons of personal interest: some mathematics educators are strongly motivated by the desire to promote productive mathematical discussions in their own classrooms (e.g. Larsen, 2013); others are strongly motivated by concerns about social justice, and are alert to inequities that might be promoted through mathematical discourse (e.g. Gutiérrez, 2013). But there is no reason why social interactions must be studied only in detail, and quantitative methods are routinely used to investigate numerous phenomena in the broader field of social studies (Neuman, 2014). We believe that diversity in methodological approaches is desirable, and that SNA might provide a focus around which researchers working primarily on social interactions and those working primarily with quantitative methods could begin productive conversations.

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MAINTAINING DRAGGING AND THE PIVOT INVARIANT IN PROCESSES OF CONJECTURE GENERATION

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In this paper, we analyze processes of conjecture generation in the context of open problems proposed in a dynamic geometry environment, when a particular dragging modality, maintaining dragging, is used. This involves dragging points while maintaining certain properties, controlling the movement of the figures. Our results suggest that the pragmatic need of physically controlling the simultaneous movements of the different parts of figures can foster the production of two chains of successive properties, hinged together by an invariant that we will call pivot invariant. Moreover, we show how the production of these chains is tied to the production of conjectures and to the processes of argumentation through which they are generated.

CONCEPTUAL FRAMEWORK AND RATIONALE

Dynamic Geometry Environments (DGEs) have acquired great interest for researchers in mathematics education and for teachers over the past years (e.g., Laborde & Strässer, 1990; Noss & Hoyles, 1996; Arzarello et al., 2002). In DGEs the figures can actually be seen and acted upon and their properties can be explored through dragging. This makes DGEs ideal for fostering, observing and analyzing processes of conjecture generation. In fact, the study of processes of conjecture generation, of argumentation and of proof in DGEs has come to be one of the leading themes of research in mathematics education (e.g., De Villiers, 1998; Hadas, Hershkowitz, & Schwarz, 2000; Arzarello et al., 2002; Mariotti, 2006).

The aim of the research we are presenting here is to yield a theoretical contribution for analyzing processes involved in conjecture generation and in argumentation within DGEs.

Dragging assumes a central role in the interaction with DGE figures, and various researchers have explored dragging modalities used by students during their explorations. In particular, Arzarello et al. (2002) and Olivero (2002) described different modalities used by the students according their goals, when generating conjectures in a DGE. Basing her work on such research, Baccaglini-Frank has identified and analysed students' use of four dragging modalities: free dragging, maintaining dragging, dragging with the trace mark active, dragging test (Baccaglini-Frank, 2010; Baccaglini-Frank & Mariotti, 2010). In this paper, we consider *maintaining dragging,* that consists in dragging some point intentionally maintaining invariant a certain property of the figure. This type of dragging is used especially in tasks that involve figuring out under which conditions certain properties are verified. In these cases, during the processes of conjecture generation a key role is played by the

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 19–26. Szeged, Hungary: PME.

solvers' perception of invariants (Baccaglini-Frank, Mariotti & Antonini, 2009; Baccaglini-Frank, 2012; Leung, Baccaglini-Frank & Mariotti, 2013), described by Neisser (1989) as "aspects of stimulus information that persist despite movements". The perception of invariants involves the sensory experience of the solver in a DGE and it is distinct from the geometrical interpretation of the objects and of their mutual relations conceived within the theory of Euclidean geometry, a process that in the literature is referred to as *discernment* (Leung et al., 2013; Leung, 2008).

With the aim of clarifying cognitive processes in a DGE, Lopez Real and Leung (2006) distinguish between the realm of DGEs and the realm of Euclidean Geometry:

> A Dynamic Geometry Environment (DGE) is a computer microworld with Euclidean geometry as the embedded infrastructure. In this computational environment, a person can evoke geometrical figures and interact with them […]. It is a virtual mathematical reality where abstract concepts and constructions can be visually reified. In particular, the traditional deductive logic and linguistic-based representation of geometrical knowledge can be re-interpreted, or even redefined, in a DGE as dynamic and process-based interactive 'motion pictures' in real time. […] There appears to be a tension (rooted in the Euclidean view of what geometry is) pulling DGE research towards the direction of bridging an experimental–theoretical gap that seems to exist between the computational microworld and the formal abstract conceptual world.
>
> (Lopez Real & Leung, 2006, pp. 665-666)

In the realm of the theory of Euclidean geometry, geometrical properties are part of a network of logical relations, each validated by a mathematical proof.

On the other hand, in the realm of a DGE, points can be moved and the figures are modified as a consequence of such induced movement. The movement of a point can be direct (when the point itself is being dragged) or indirect (if the movement is a consequence of the dragging of another point): in this second case, the movement of the dragged point causes the movement of other points (because all the elements of the figure have to maintain the logical relationships imposed by the construction steps). The invariants are perceived in space but also in time and properties can be perceived simultaneously or in distinct temporal instances.

CONTROLLING THE DGE FIGURE

To ease the reading of this paper we introduce the theoretical notions referring to one of the tasks assigned during the study.

Task: Construct: a point P and a line *r* through P, the perpendicular line to *r* through P, C on the perpendicular line, a point A symmetric to C with respect to P, a point D on the side of *r* containing A, the circle with center C and radius CP, point B as the second intersection between the circle and the line through P and D. Make conjectures about the possible types of quadrilateral ABCD can become, describing all the ways you can obtain a particular type of quadrilateral.

The figure resulting from the construction (Fig. 1) can be acted upon by dragging points (in our example we can think about dragging D), and some properties can be recognized as invariants for any movement of the dragged point (e.g., “CP = PA”).

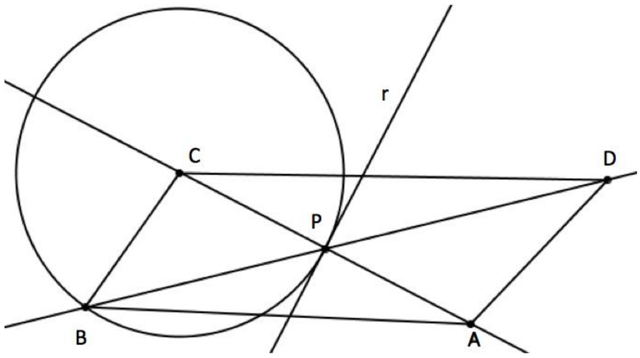

Figure 1: a possible result of the construction in the task above.

As the solver acts on the figure with the aim of generating a conjecture, s/he can decide to intentionally induce an invariant by dragging a point (e.g., “ABCD parallelogram” by dragging D), performing *maintaining dragging*.

The use of maintaining dragging involves dragging a point maintaining certain properties as invariants. Moving a point so that a DGE figure maintains a certain property requires a high level of control over the movement of different parts of the figure. The solver has to manage the relationships between the movements of the different parts of the figure (a similar case of coordinating movements has been studied in relation to cognitive processes involved in the use of pantographs, see Martignone & Antonini, 2009). The solver usually exercises indirect control over the invariant to maintain: its movement depends on the movement of the dragged point, that the solver controls directly, and sometimes the movements of the different parts of the figure are difficult to coordinate. The reader can experience the difficulty by trying to drag D (as in Fig. 1) maintaining the property “ABCD parallelogram”.

To maintain an interesting configuration (A) the solver needs to control the figure more directly, so s/he passes from property A to a property A_1, which is easier and more direct to control, and such that its presence guarantees the presence of A. The nature of this process is abductive, similar to that described in Arzarello et al. (2002), were for *abduction* the following type of inference is intended: (*fact*) a fact A is observed; (*rule*) if C were true, then A would certainly be true; (*hypothesis*) so, it is reasonable to assume C is true (Pierce, 1960). In our case, given A_1, a property A_2 is generated such that $A_2 \Rightarrow A_1$. The spark initiating this abductive process is the need of better controlling the movements of the figure. Iterating the process may lead to an abductive chain of properties A_1, A_2, …A_n, such that each property ensures better control over the desired configuration A. This chain can later, in the proving phase, be flipped into a chain of deductive implications ($A_n \Rightarrow \ldots \Rightarrow A_2 \Rightarrow A_1$). So the spark initiating the successive inferences finds its origin in the pragmatic need of controlling the figure, and in particular in the need to coordinate the movement of the dragged point with that of the other parts of the figure in order to maintain invariant a desired property.

We stress how the search for a logical relation within the theory of Euclidean geometry sparks from an experience within the realm of the DGE. The solver will seek for fragments of theory with the goal of ameliorating his/her haptic control over the figure. These same fragments of theory can later be used for constructing an argumentation and finally a proof for the conjecture s/he will have reached.

The pivot invariant

During a first phase of the exploration leading to a conjecture, the chain $A_n \Rightarrow \ldots \Rightarrow A_2 \Rightarrow A_1 \Rightarrow A$ is developed as the solver searches for invariants to control more easily during maintaining dragging. When maintaining dragging is used, in a second phase, a new invariant B_1 can be perceived simultaneously, as A_n is maintained. For the solver this new property B_1 has a very different status than the properties A_i. First of all, B_1 can be controlled directly. Secondly, the relation between the properties B_1 and A_n is frequently perceived as causal (hence the arrow "$\rightarrow$" and not the implication symbol "$\Rightarrow$" in Figure 2) in the realm of DGE: the presence of B_1 guarantees (visually, for now) the simultaneous presence of A_n. Later, this causal relation can be interpreted logically as the implication $B_1 \Rightarrow A_n$. The process of conjecture generation may continue, leading to a second invariant B_2 simultaneously perceived during dragging, a third one B_3, etc., up to the generation of B_m . These geometrical properties form a new chain of relationships perceived as causal relations in the realm of the DGE (Figure 2). The invariant A_n acts as a pivot between the two chains of invariants and plays a fundamental role in the development of a conditional link between the properties that become the premise (B_m) and the conclusion (A) of the conjecture generated. We call A_n the *pivot invariant*.

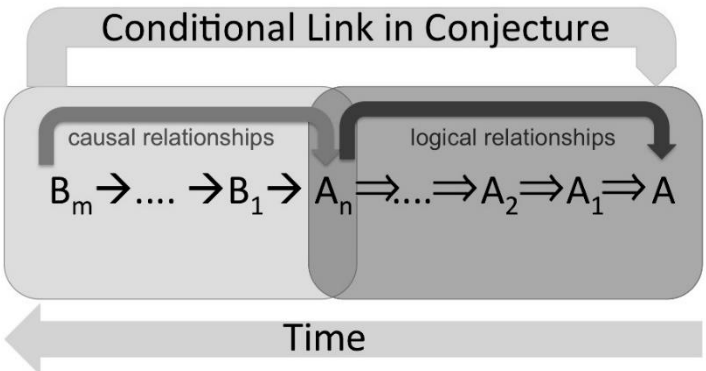

Figure 2: the two chains of invariants linked by the pivot invariant, discovered during the exploration that proceeds in time from right to left.

THE PIVOT INVARIANT AT WORK: ANALYSIS OF A CASE

To show how the notion of pivot invariant can bring insight to students' conjecturing activity in a DGE we now present excerpts from an interview conducted with two students, Ste and Giu, both in the second year of high school (15 and 16 years old, respectively) in a northern Italian Liceo Scientifico. They had used Cabri II Plus the year before and they had been introduced to the four types of dragging introduced earlier in this paper (free dragging, maintaining dragging, dragging with the trace mark active, dragging test) during two previous lessons (for a complete description of the study see Baccaglini-Frank, 2010, or Baccaglini-Frank & Mariotti, 2010). The excerpts we present are from the part of the interview on the problem presented as an example in the previous section of this paper. The students' exploration lasted approximately 20 minutes, from the construction of the figure to the writing of their first conjecture on the possibility of obtaining a parallelogram. We analyze the excerpt in which the students identify the parallelogram as a possible configuration and decide to try to maintain it (the name of the student holding the mouse is in bold).

Giu: …So to get a parallelogram, what it looked like in the beginning…

Ste: So BP = PD by definition, that we know for sure.

Giu: Yes, yes, because they intersect at their midpoints, they are the diagonals.

Ste and Giu write a first conjecture: "ABCD is a parallelogram when BP = PD (that is when P is the midpoint of BD)". As Ste writes, Giu thinks aloud and proposes a proof for the conjecture.

Giu: Why? Because when this here is a diagonal [BD] of the parallel… of the quadrilateral, this here is another diagonal [CA] of the parall…of the quadrilateral.

Ste: But I need to add [referring to the written conjecture] in a parenthesis that CP is equal to PA.

Giu: CP = PA by definition, BP is equal to PD because we said so…and so they are diagonals that intersect at their midpoints, so it is a parallelogram.

The students have produced the following chain of deductions: BP = PD (A_2) $\Rightarrow$ diagonals intersect at their midpoints (A_1) $\Rightarrow$ ABCD is a parallelogram (A).

At this point the students start searching for ways of acting upon the figure to maintain property A during dragging. They seem to be seeking for ways to better control the movement of the different parts of the figure, that they still seem to find difficult to control through A_2. Suddenly Giu constructs the circle C_{PD} (a circle with center in P and radius PD, see the Fig. 3) and says:

Giu: But see, you can do it like this. You can see that like this is comes out only when…no, you see…[he drags D so that C_{PD} passes through the intersection, defined as B, between the line PD and the circle C_{CP}] […] You try and see *so that* this thing [concurrence of C_{PD}, C_{CP} and t]…is maintained. (Figure 3)

Ste takes back the mouse and the students' attention shifts to the passage of C_{PD} through B, the intersection between the line t (through P and D) and C_{CP} (Figure 3).

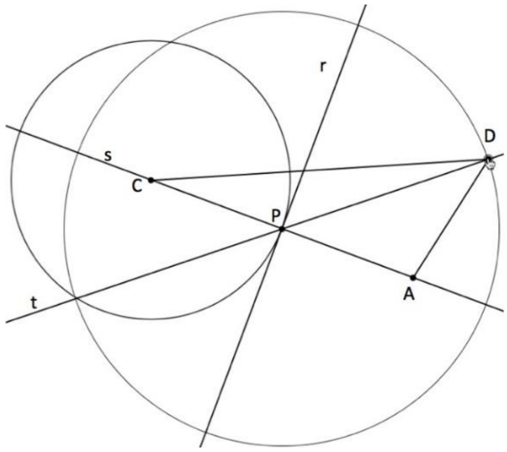

Figure 3: The students perform maintaining dragging inducing C_{PD} to pass through the intersection of C_{CP} with line *t*, defined as B in the original construction.

This is a new property A_3 that the students try to induce as an invariant. This property has been inferred through an abduction:

(*facts*) BP = PD (A_2), B is the second intersection of *t* with C_{CP}.

(*rule*) If B lies on C_{PD}, then BP = PD.

(hypothesis) B lies on C_{PD} (A_3).

Dragging D it seems to be easier for the students to control A_3 than A_2. This is a fundamental moment in the process of conjecture generation: the invariant A_3 is discerned through an abduction in order to better control the figure and it is used for performing maintaining dragging in search for new invariants that might be causing A_3 (and the chain of invariants $A_3 \Rightarrow A_2 \Rightarrow A_1 \Rightarrow A$) to be visually verified. Elements of different natures concur in the discernment of this invariant: some are theoretical (e.g., the "rule" in the abduction is a theorem of Euclidean geometry) and others related to the phenomenology of the DGE (also see Leung et al., 2013).

As Giu continues to explore "when" A_3 is visually verified, he asks Ste to take back the mouse to concentrate on the movement of D when the trace is activated on it.

Giu: You maintain these things [B on C_{PD}]…it looks like a curve.

Ste: It's really hard!

Giu: Yes, I know…I can only imagine. It looks like a circle…with center in A.

Ste: It has to necessarily have radius AD! Anyway you would need AP to equal AD [he holds the mouse but stops dragging].

Ste is concentrated on maintaining the invariant A_3 while Giu tries to geometrically describe the trace mark. The students discern two invariants during dragging: $D \in C_{AP}$ (B_1), and PA = AD (B_2). Once the students construct C_{AP}, perform a dragging test dragging D along it, and notice that ABCD does seem to remain a parallelogram in this case, they write their final conjecture: "ABCD is a parallelogram if PA = AD" ($B_2 \Rightarrow$ A).

The invariants A_i are interpreted as logical consequences of the invariants B_j, and they are explicitly linked to them through the invariant A_3, which is a pivot-invariant. While the invariants A_i arise mostly thanks to the students' theoretical knowledge of Euclidean geometry, the invariants $B_1, \ldots, B_m$ appear thanks to support offered within the phenomenological domain of the DGE, where invariants can be perceived through simultaneous perception accompanied by different levels of pragmatic control over the varying parts of the figure. The theoretical and pragmatic domain are hinged together by the pivot invariant that comes to life with hybrid characteristics.

CONCLUSIONS

The analysis provided in this paper clarifies a specific process of conjecture generation in a DGE. In particular, the pragmatic need of physically controlling the figures explains how, in certain cases, the search for logical relations in the theory of Euclidean geometry can be fostered, together with the production of a chain of abductions leading to the conjecture. This new way of explaining how students come to substitute an invariant to maintain with a new property generalizes and ameliorates earlier descriptions of the process (see Baccaglini-Frank, 2010; Baccaglini-Frank & Mariotti, 2010): the process is now explained through the necessity of better controlling the figure, which leads to the production of two abductive chains.

The study points to at least two new directions of research. One is theoretical: the two chains of inferences may not necessarily be produced one after the other. In general, the two chains might be intertwined, leading to a greater complexity that needs to be further investigated. The second direction is practical and involves teachers. We believe that the theoretical notion of pivot invariant could be useful for a teacher who decides to promote students' conjecture generation in a DGE. Indeed, s/he could use it to gain deeper insight into students' processes of conjecture generation, and thus to better guide students' processes of conjecturing, argumentation, and proof.

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DOES STUDYING LOGIC IMPROVE LOGICAL REASONING?

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There has long been debate over whether studying mathematics improves one's logical reasoning skills. In fact, it is even unclear whether studying logic improves one's logical reasoning skills. A previous study found no improvement in conditional reasoning behaviour in students taking a semester long course in logic. However, the reasoning task employed in that study has since been criticised, and may not be a valid measure of reasoning. Here, we investigated the development of abstract conditional reasoning skills in students taking a course in formal logic, using a more sophisticated measure. Students who had previous experience of logic improved significantly, while students with no previous experience did not improve. Our results suggest that it is possible to teach logical thinking, given a certain degree of exposure.

INTRODUCTION

Since the time of Plato (375B.C./2003) it has been assumed that people can be taught to think more logically, and in particular, that mathematics is a useful tool for doing so. This is known as the Theory of Formal Discipline (TFD) and is exemplified by the philosopher John Locke's suggestion that mathematics ought to be taught to "all those who have time and opportunity, not so much to make them mathematicians as to make them reasonable creatures" (Locke, 1706/1971, p.20). While there is some evidence that studying mathematics does indeed improve logical thinking skills, there is little evidence that studying logic itself improves one's logical thinking. The aim of the current study was to investigate the development of logical reasoning skills in undergraduate students taking a course in introductory formal logic. Before describing our study, we begin by reviewing the evidence that studying mathematics improves reasoning skills, then review previous investigations of whether studying logic can improve reasoning skills, along with the flaws in these investigations that we aimed to remedy.

The TFD was first tested systematically by Thorndike (1924), who measured children's general reasoning skills before and after one year of schooling. He reported that the subjects students studied had only a minimal influence on changes to their test scores. French, chemistry and trigonometry were associated with the largest, yet small, improvements, while other areas of mathematics (arithmetic, geometry and algebra) were associated with improvements close to zero.

However, Lehman and Nisbett (1990) found evidence that studying mathematics at university level was associated with improved conditional reasoning skills. Reasoning about conditional 'if…then' statements is a central component of logical reasoning (Inglis & Simpson, 2008), and fundamental to mathematics (Polya, 1954). Lehman and Nisbett tested US undergraduates in their first and fourth years of study on statistical

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 27–34. Szeged, Hungary: PME.

and methodological reasoning, conditional reasoning and verbal reasoning. Across their whole sample, which was formed of students from several majors, they found a correlation between number of mathematics courses taken and change in conditional reasoning behaviour (r = .31). The correlation was even stronger within the natural science majors (r = .66). This is a promising finding which suggests that conditional reasoning is an aspect of logical thinking that can be developed through mathematics education.

Conditional reasoning development was also investigated by Inglis and Simpson (2009), who compared scores in mathematics and non-mathematics undergraduates on entry to a UK university. They gave the undergraduates an abstract Conditional Inference Task which involved judging the validity of conclusions drawn from abstract conditional statements (e.g. If the letter is D then the number is 3; the number is not 3; conclusion: the letter is not D). Mathematics undergraduates performed significantly better than the comparison undergraduates, even after controlling for between-group differences in an intelligence measure. However, over the course of a year, the mathematics students improved by only 1.8%, which was not significant.

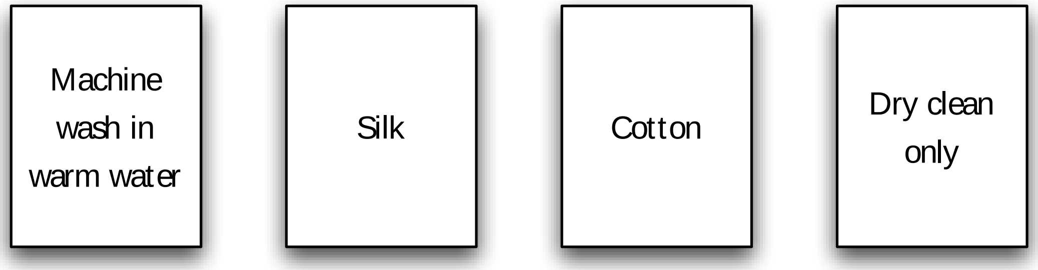
In a similar study, Attridge & Inglis (2013) investigated the development of conditional reasoning skills in mathematics and non-mathematics A-level students (A-levels are two-year post-compulsory courses in the UK, the results of which are used by universities to select incoming undergraduates). There was no difference between groups in conditional reasoning at the beginning of A-levels, but after one year the mathematics students' reasoning had significantly improved whereas the non-mathematics students' reasoning had not.

While there is evidence that studying mathematics at A-level (Attridge & Inglis, 2013) and undergraduate level (Lehman & Nisbett, 1990) is associated with improved conditional reasoning skills, there is less evidence that studying logic itself is associated with improvements in logical thinking. Next we review two studies that investigated this question and came to different conclusions.

Cheng, Holyoak, Nisbett and Oliver (1986) investigated the development of conditional reasoning skills in undergraduates taking a semester-long course in logic. The students completed four Wason Selection Tasks (with a mixture of conditional and biconditional statements and abstract and thematic content, see Figure 1 for an example) at the beginning and end of the course, which contained 40 hours of teaching, including the definition of the conditional. It seems reasonable to expect that after such training students should be fairly competent at dealing with conditional statements; it is difficult to imagine a more promising way to improve a student's logical thinking competency. Nonetheless, there was a non-significant decrease in errors of only 3%.

However, the lack of improvement that Cheng et al (1986) observed could be due to the measure they used. Since their study was conducted it has been suggested that Selection Tasks may not actually measure conditional reasoning skills, particularly

As part of your job as quality control inspector at a shirt factory, you have the task of checking fabric and washing instruction labels to make sure they are correctly paired. Fabric and washing instruction labels are sewn back to back. Your task is to make sure that all silk labels have the 'dry clean only' label on the other side.



You must only turn over those labels you need to check to make sure the labels are correct.

Figure 1. An example Wason Selection Task similar to those used by Cheng, Holyoak, Nisbett and Oliver (1986).

when the task is presented with a real-world context (Sperber, Cara & Girotto, 1995; Sperber & Girotto, 2002). Sperber et al. suggested that Selection Task performance is highly influenced by contextual judgments that pre-empt any reasoning. Their account, which implies that Selection Tasks do not actually measure reasoning processes, was supported across six studies (Sperber et al., 1995; Sperber & Girotto, 2002). Sperber and his colleagues showed that success rates in the task can be dramatically manipulated by altering the relevance of the content. Success in descriptive versions of the task can be increased to over 50%, in line with the success rates usually found with obligation-based contextual versions (Sperber et al., 1995).

Given that their measure may not actually reflect reasoning processes, Cheng et al's (1986) results are difficult to interpret. It may be that their participants did improve in logical reasoning, and this simply wasn't reflected in their measure. This interpretation is consistent with a similar study on teaching reasoning, in which White (1936) investigated the effect of logic training on 12-year-old boys' reasoning ability. One class spent an hour per week for three months being taught logic, including deduction, induction and syllogisms, while another class were not taught any logic. At the end of the three months the students were given a reasoning test that included, among other things, syllogism validity judgments. The class that had been taught logic scored significantly higher on the reasoning test than the control class. The authors concluded, conversely to Cheng et al. (1986), that logical thinking can be taught.

The difference in findings between White (1936) and Cheng et al (1986) may be due to the difference in the reasoning measure used, or it may be due to the difference in age between the participants in the two studies. Perhaps 12-year-olds' reasoning skills are more malleable than undergraduates' reasoning skills. To distinguish between these possibilities, we investigated reasoning development in undergraduates studying introductory formal logic, using a more sophisticated measure than a Selection Task. Undergraduate students completed an abstract conditional inference task (based on Attridge and Inglis' (2013) finding that studying mathematics was associated with improvement on an abstract conditional inference task) before and after being introduced to truth-functional logic.

METHOD

Design

The study followed a one-group pre-test/post-test design where the intervention was a course in logic and the pre-test and post-test was an abstract conditional reasoning task.

Participants

Participants (60 males, 19 females) were undergraduate students taking a course on logic at a medium-sized private research university in the South-Eastern United States. Students came from various majors, including computer science, software engineering, mechanical engineering, aerospace engineering, physics, and business. At Time 1, 79 participants completed the test and of these, 58 also completed it at Time 2.

Materials

To measure logical reasoning, we administered the abstract conditional inference task (Evans, Clibbens & Rood, 1995). In this task, participants are given a conditional rule (e.g. If the letter is M then the number is 5) along with a premise about that rule (e.g. The letter is M), followed by a conclusion derived from the rule and premise (e.g. The number is 5). The participant then deduces whether the inference to the conclusion is necessarily valid or invalid. The task contains 16 items of four inference types: Modus Ponens (MP; if p then q, p, therefore q), Denial of the Antecedent (DA; if p then q, not-p, therefore not-q), Affirmation of the Consequent (AC; if p then q, q, therefore p) and Modus Tollens (MT; if p then q, not-q, therefore not-p). The lexical content of the rules (letters and numbers) was generated randomly and the order of the problems was randomised by participant. The instructions were adapted from Evans et al.

Logic course

The course consisted of 37.5 hours of lectures over 15 weeks, covering traditional logic, symbolic logic and informal logic. The assessment consisted of 14 pop quizzes, two mid-term exams and a final exam. The participants were taught in three groups.

Procedure

Participants completed the tests in class at the beginning of the course in early January 2014, and again at the end of the course in late April 2014. Tests were completed using pen and paper under exam-style conditions.

RESULTS

Participants were split into two groups depending on whether or not they had previous experience with logic. This was determined on the basis of each participant's degree programme, and whether that programme usually involved a course with some degree of logic content prior to the one in question here. As such, this is only a proxy for prior logic experience. The prior logic group comprised students majoring in Computer Science, Software Engineering, or Electrical Engineering, all of whom should have taken Digital Logic or Discrete Mathematics before they take Logic. This allowed us to investigate the role of previous exposure to logic in any development found. One participant was removed on the basis of being an outlier in terms of change over time (scoring 16/16 at Time 1 and 5/16 at Time 2). The remaining 57 participants' data were subjected to a 2 (Time: pre-test, post-test) × 4 (Inference: MP, DA, AC, MT) × 2 (Prior Logic: yes, no) mixed ANOVA.

This revealed a main effect of inference, $F(3,165) = 41.31$, $p < .001$, $\eta_p^2 = .429$, where accuracy was higher on MP inferences than on all other inferences, all *p*s < .001, a main effect of Time $F(1,55) = 6.78$, $p = .012$, $\eta_p^2 = .110$, with higher accuracy at Time 2 ($M = 2.70$, $SD = 0.79$) than at Time 1 ($M = 2.54$, $SD = 0.63$), and no main effect of Prior Logic, $F(1,55) = 1.77$, $p = .188$, $\eta_p^2 = .031$. However, there was a significant interaction between Time and Prior Logic, $F(1,55) = 4.32$, $p = .042$, $\eta_p^2 = .073$ (see Figure 2). An independent samples t-test showed no difference in Time 1 scores between participants with ($M = 2.54$, $SD = 0.61$) and without ($M = 2.48$, $SD = 0.58$) prior logic experience, $t(55) = .37$, $p = .716$, $d = 0.1$. However, paired samples t-tests showed that in the students presumed to have studied logic previously, scores significantly improved between Time 1 ($M = 2.54$, $SD = 0.61$) and Time 2 ($M = 2.91$, $SD = 0.87$), $t(32) = 3.27$, $p = .003$, $d = 0.49$, while in the students who had not studied logic previously, scores did not significantly improve between Time 1 ($M = 2.48$, $SD = 0.58$) and Time 2 ($M = 2.52$, $SD = 0.63$), $t(23) = .41$, $p = .682$, $d = .07$. Despite this, the difference between groups at Time 2 was only marginally significant, $t(55) = 1.97$, $p = .054$, $d = 0.53$. All other interactions were non-significant.

	Average	Time 1	Time 2	Absolute change	Percentage change
MP	3.85(0.35)	3.77 (0.63)	3.93 (0.26)	+0.16 (0.65)	+4.24
DA	2.26 (0.98)	2.12 (1.46)	2.40 (1.47)	+0.28 (1.36)	+13.21
AC	1.98 (1.43)	1.91 (1.61)	2.05 (1.57)	+0.14 (1.38)	+7.33
MT	2.42 (0.87)	2.25 (1.30)	2.60 (1.22)	+0.35 (1.84)	+15.56

Table 1. Mean Conditional Inference Scores split by Time and Inference. Standard deviations in parentheses.

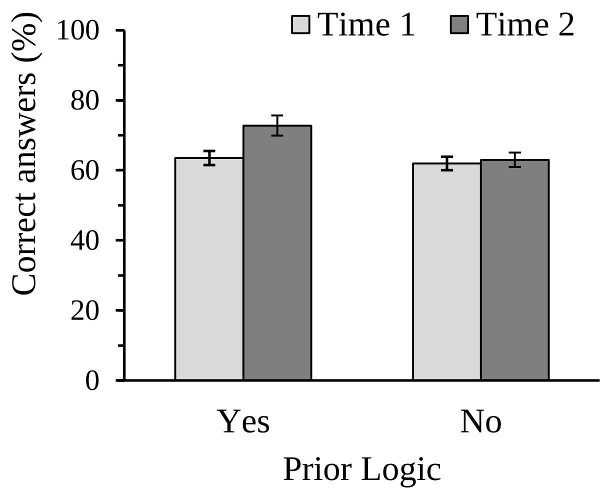


Figure 2. Interaction between Time and Prior Logic on Conditional Inference Scores. Error bars reflect ±1 standard error of the mean.

DISCUSSION

We investigated the development of conditional reasoning skills in undergraduates taking a course in logic. Overall, our results suggest that studying formal logic improves students' ability to deal with conditional statements, but only if they have had some experience with logic previously. While conditional inference scores did improve over time for the whole sample, when we examined the role of previous experience with logic, it became apparent that only those who had studied logic previously actually showed any gains in reasoning skills during the course. For those students who had not studied logic before, there was not a significant improvement in conditional inference scores over time. Interestingly, the students who had taken a logic course previously did not outperform those who had not at Time 1. This suggests that the amount of logic training the students had received previously was not sufficient to give them an advantage on our conditional inference task, but that it was sufficient to make the logic course in question more effective.

Our findings suggest that it is possible to teach logical thinking, but that a certain level of exposure may be necessary before students' skills begin to develop. We do not have data on the number of hours of previous study that participants had, but the fact that students without prior experience did not improve during the 37.5 hours of lectures involved in the current course suggests that a greater number of hours is required for development. Future research should systematically investigate the number of hours of exposure necessary for students' logical reasoning skills to improve.

It is interesting to note that the improvement we saw in conditional reasoning did not differ between the four inference types (MP, DA, AC and MT). Attridge and Inglis (2013) found that studying A level mathematics was associated with improved performance with the invalid inferences (DA and AC), and with *worse* performance on the MT inferences. In the present study, students improved to a similar extent on all of the inferences, which, perhaps unsurprisingly, suggests that teaching students logic is a more effective way to improve their logical thinking than teaching them mathematics. Nevertheless, if we compare the effect sizes for the increase in the number of correct responses, over all inference types, in each of these studies, the A level mathematics students' improvement (d = .49) was of a similar magnitude to that of the logic students who had some prior exposure to logic (d = .49) and larger than in the full logic class sample (d = .34). This was despite the mathematics students having no previous experience with logic and not receiving any explicit logic tuition during their A level. On the other hand, the mathematics course lasted for a full academic year, as opposed to one semester for the logic course. Although the two courses are not comparable in terms of length or student age and experience, the fact that learning mathematics appears to develop one's logical reasoning skills to a similar extent to studying formal logic is very promising for proponents of the TFD.

Our results contradict those of Cheng et al (1986) who found that a semester long course in logic was not associated with any improvements in students' reasoning performance. We suggested that the measure Cheng et al used, four selection tasks, was not an appropriate measure of reasoning, and that this may be why they failed to find an effect of tuition. Our results support this interpretation: using a more sophisticated measure of conditional reasoning we found that a similar intervention resulted in significant improvement.

One limitation of our study is that we did not compare the logic students to a control group. This means that we cannot rule out the possibility that our participants would have improved even without taking the logic course. However, this alternative interpretation seems unlikely. First, the improvement was only seen in the subset of students with prior exposure to logic. If there were a general developmental trend in reasoning skills in the undergraduate population then we would expect to see this development across the whole sample. Second, Attridge (2013) did not observe any development in conditional reasoning skills in a sample of psychology undergraduates, and Inglis and Simpson (2009) did not observe any improvement in undergraduate mathematics students. Again, if the development we observed here were due to a general developmental trend, as opposed to the logic course, we would expect to have seen improvements in both of these groups.

Another limitation is that we did not directly measure prior logic experience; we used each participant's major as a proxy for whether or not they were likely to have taken a course with some logic content previously. This means that a few students in each group could have been miscategorised. Since we split participants by major, there is also the possibility that participants in the prior logic and non-prior logic groups may have varied on SAT scores or another unmeasured variable. However, there was an overall effect of time on conditional inference scores, averaging over both groups, so these issues should not be a major cause for concern. Rather, the effect of prior logic experience should be confirmed in future studies where potential confounding variables are controlled for, and if it transpires that a factor such as SAT scores is responsible for the group difference, then this in itself would be an interesting finding.

In conclusion, our findings suggest that, contrary to previous research, it is possible to improve students' logical reasoning through instruction. Nevertheless, the level of improvement we found was comparable to that seen in A level mathematics students, who received no explicit logic tuition. This is promising for proponents of the TFD, which suggests that teaching mathematics is an effective method for developing students' logical thinking skills.

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MULTIPLICATION AND DIVISION PROBLEMS POSED BY PRE-SERVICE ELEMENTARY MATHEMATICS TEACHERS ABOUT FRACTION TOPIC

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The purpose of this study is to analyze the fraction multiplication and division problems posed by pre-service elementary mathematics teachers. A total of 213 pre-service teachers enrolled in different years of Elementary Mathematics Teacher Education program at a state university in the eastern part of Turkey took part in the study. Two fraction operation (1 multiplication and 1 division) is used as the data collection tool in the study. With a view to finding out the types of problems posed by the pre-service teachers, the data were analyzed using qualitative descriptive analysis. The findings suggest that, pre-service teachers have difficulties in posing problems regarding multiplication and division. This indicates that a conceptual understanding has yet to be achieved.

INTRODUCTION

The rational numbers include, in addition to natural numbers, fractions and decimals. The fractions, in turn constitute a fundamental element of decimals, rational numbers, ratio and proportion, and measurement systems. In this context, rational numbers, and therefore fractions are crucial elements of our daily lives. Fractions are rich as a field of mathematics, complicated as a cognitive object, and among the more difficult mathematical concepts to be taught and to teach (Smith, 2002). That is why the concept of fractions and operations using fractions are in the lead of the topics the students have difficulty in mastering (Ma, 1999; Tirosh, 2000; Yim, 2010; Zembat, 2007). The difficulties encountered in the teaching of fractions were investigated in numerous studies (Haser & Ubuz, 2002). The studies reveal that students of all grades have a hard time with fractions as well as in solving and posing fraction problems (Kocaoğlu & Yenilmez, 2010).

Problem posing

Problem solving and posing skills are among the most fundamental topics in mathematics curriculums (Abramovich, 2014; Chen, Dooren, Chen & Verschaffel, 2011). Problem solving is a process and from a mathematical perspective, it is about getting the sole correct answer (Kojima, Miwa & Matsui, 2015). For the students to be able to develop a problem solving skill, they should first have a problem posing skill in place (Turhan & Güven, 2014). Problem posing, is essentially a problem solving activity entailing the development of questions and new problems to be discovered or analyzed about a given case (Akay, Soybaş & Argün, 2006), and it requires creative thinking between multiple answers (Kojima, Miwa & Matsui, 2015). Many studies emphasize the significant contributions problem posing activities has on the development of the students (Kılıç, 2013; Toluk-Uçar, 2009; Turhan & Güven, 2014). Problem posing enables teaching of mathematical reasoning, and instills the abilities to discover mathematical cases and to formally describe mathematical cases in speech and text. However, problem posing skills can be developed only where the teachers design appropriate learning and teaching processes (Akay, Soybaş & Argün, 2006).

Fractions

The subject of fractions lies at the core of many topics in mathematics (Behr et al., 1992; Kieren, 1976). But it's difficult for students to learn rational numbers and relevant concepts. Students have significant difficulties in topics such as ordering fractions, and addition, subtraction, multiplication, and problems with fractions (Soylu & Soylu, 2005). Teachers, as well as pre-service teachers are also found to experience such difficulties with rational numbers (Redmond, 2009; Tirosh, 2000; Toluk-Uçar, 2009; Yim, 2010). These difficulties affect the problem posing process.

Studies suggest that when asked to pose a problem involving divisions with fractions, teachers and pre-service teachers usually pose multiplication problems, or fail to correctly pose one at all (Tirosh, 2000). Ball (1990) reported that all the pre-service teachers that participated in the study successfully calculated the expression $1\frac{3}{4}:\frac{1}{2}$, but most of them could not pose a verbal problems that describe this expression. Similar results were reported by Toluk-Uçar (2009). Işık (2011) focused on the conceptual analysis of the problems posed by pre-service elementary mathematics teachers, with respect to division with fractions. Işık (2011) indicated that pre-service teachers usually omitted the measurement purpose of the division, which suggested that the conceptual structure of division with fractions was not established properly in the problems.

A few studies touch briefly on the common errors regarding multiplication with fractions. Toluk-Uçar (2009), in particular, noted that when asked to pose a verbal problem to reflect the operation $\frac{3}{4}x\frac{1}{3}$, pre-service primary teachers usually posed a problem reflecting the operation $\frac{3}{4}:3$. Moreover, Işık (2011) found that pre-service elementary mathematics teachers were usually successful in attaching meanings to operations and numbers in the problems posed for multiplication with fractions. He noted that pre-service teachers had more conceptual difficulties in problem posing for divisions with fractions, compared to multiplication with fractions (Işık, 2011).

The studies about problem posing indicate that, pre-service teachers' level of success in the problem posing skills is generally low (Işık, 2011; Toluk-Uçar, 2009; Zembat, 2007). Instilling mathematics problem posing skills provide many benefits to teachers and pre-service teachers. For instance, in cases where the problems offered in textbooks are insufficient, or are inappropriate for the skills of the students, or do not reflect the interests and needs of the students, the teacher would be required to pose original problems regarding the topic, to further teaching (Korkmaz & Gür, 2006). Accordingly, the purpose of the present study is to provide a conceptual analysis of the problems posed with respect to multiplication and division with fractions, by pre-service elementary mathematics teachers in different years of a teacher training program, and investigates the variation of such problems with reference to the year of training.

METHODOLOGY

This study offers a cross-sectional study of developmental research methods, which are usually characterized by their aim to compare, define, categorize, and analyze the individuals, groups, organizations, methods, or materials, to gain insight into their differences, and to interpret the results of the analysis (Miller, 1998). Therefore the study group comprises 213 (47 are in first, 53 are in second, 57 are in third, and 56 are in fourth year of school) pre-service elementary mathematics teachers from various years of teacher training, at the Department of Elementary Mathematics Education in a state university, in Turkey.

The data collection tool used in the study is 2 operations, 1 multiplication and 1 division, as was the case with Işık's study (2011). The first operation is $\frac{1}{2}x\frac{1}{8}$ and the second operation is $\frac{1}{2}:\frac{1}{10}$. The multiplication and division operations involve the multiplication or division of two proper fractions one of which is a half. These operations are offered to each pre-service teacher in written form.

The problems posed by the elementary pre-service teachers were analyzed using the qualitative descriptive analysis method, with reference to the problem types identified by Işık (2011). Each problem sentence posed by pre-service teachers was read and reviewed carefully, and a classification was sought by coding the sentences. Throughout the process, the problems were analyzed by the author, and two Ph.D. students. The analyses by the author and the researchers were found to be consistent at a rate of 91%. Thereafter, the analyses by the author and the researchers were compared and at the end the categorization of each and every problem posed by the respondents was finished.

FINDINGS

In this section, the categories of problems employed by the pre-service mathematics teachers for multiplication will be presented in separate tables for two expressions, along with percentage and frequency figures.

The categories employed by pre-service elementary mathematics teachers in formulating a problem to verbalize the multiplication of a proper fraction that is equal to one half, with another –in this case, 1/8–, are shown in Table 1.

Categories \ Grades	1st year (n=47) %	2nd year (n=53) %	3rd year (n=57) %	4th year (n=56) %	Total (n=213) %
Finding out 1/8 of the half of a whole	38	30	28	23	30
To calculate 1/8 of 1/2 of the whole	4	6	12	5	7
Simple exercises	17	19	14	35	15
To calculate 1/8 of the half of a plurality	11	-	5	2	4
Through division	15	36	37	21	28
Incorrect statements	7	9	4	14	9
Empty	5	-	-	-	1
Total	100	100	100	100	100

Table 1: The categories of the problems posed by pre-service teachers for “1/2 x 1/8”

The largest group (38%) of first year pre-service teachers posed problems to find 1/8 of the half of the whole. An example of this category of problems is: “An athlete ran 1/8 of the half of the full track. What fraction of the full track did he run?” First years’ 17% chose simple exercises, and another 15% posed problems which could be solved through divisions. Posing of problems to calculate 1/8 of the half of a plurality was the choice of 11%. Moreover, 6% of the responded supplied incorrect posing, and a further 6% supplied none. Thirty six percentage of the sophomores posed problems based on division. For instance: “8 friends want to share half an apple. What fraction of an apple would each get?” Whereas 30% posed problems to find 1/8 of the half of the whole. Nine percentage of second year respondents posed problems containing incorrect statements and 19% posed simple exercises. Simple exercises are usually posed as “What would be the result if we multiplied 1/2 with 1/8?” In case of third year students, the problems based on divisions constitute the most frequent category (37%), followed by problems posed to calculate 1/8 of the half of the whole (28%), to calculate 1/8 of 1/2 of a plurality (12%), and simple exercises (14%). Senior years opted for simple exercises (35%), calculation of 1/8 of the half of the whole (23%), and problems based on division (21%). At the same time 14% of seniors posed problems containing incorrect statements. For instance, “Kaya ate 1/2 of his cake. Then he ate a further 1/8. What portion of the cake did he eat?”

The categories employed by the pre-service teachers vary by year. While freshmen devise problems to calculate 1/8 of the half of the whole, 2nd and 3rd year students had a preference for problems based on divisions. Fourth year students, on the other hand, mostly posed simple exercises. The question was skipped only by some freshmen, while no 2nd, 3rd, or 4th year student skipped the question. Furthermore, the problems requiring the application of proportions were posed only by juniors.

The problems regarding the division of a proper fraction equal to one half by another proper fraction, posed by pre-service elementary mathematics teachers are summarized in Table 2.

Categories \ Grades	1st year (n=47) %	2nd year (n=53) %	3rd year (n=57) %	4th year (n=56) %	Total (n=213) %
Measurement	2	33	14	12	16
Proportion	-	8	7	4	5
Simple exercises	6	15	9	22	12
Through multiplication	4	17	9	25	14
Confused multiplication-based operations	28	8	26	30	23
Incorrect statements	6	15	9	7	10
Empty	54	4	26	-	20
Total	100	100	100	100	100

Table 2: The categories of the problems posed by pre-service teachers for "1/2 : 1/10"

More than half (54%) of the first year pre-service elementary mathematics teachers skipped the question, while 28% posed problems in the category of confusion with multiplication. Take a look at the following example: "Ali's mom saved half of the cake for him to eat later. As Ali is able to eat one tenth of the cake, what fraction of the cake is gone?" This problem requires multiplication, rather than division. Furthermore, 6% of the freshmen posed problems with incorrect statements, whereas a further 6% posed simple exercises such as "What is the result of the division of one half by 1/10?" 33% of the second year pre-service teachers posed problems in the measurement category; 17% in the through multiplication category; such as "Ayşe wants to eat 10 times the half of the cake, and save the rest for her sister. What fraction of the cake should Ayşe eat?" which can be solved through multiplication; and 15% in the simple exercises category. Problems posed with incorrect statements were offered by 13%. An example of these problems is as follows: "What part of the whole would we get by dividing half a bread by 1/10?" Furthermore, 8% of sophomores posed problems based on proportions, such as "What is the proportion of the half of a given number, over the one tenth of the same number?" While the problems posed by a second 8% had confusions with multiplication. Twenty six percentage of the third year participants failed to pose a problem and skipped the question, while 26% had confusions with multiplication. Measurement-related problems were posed by 14%, whereas through multiplication problems were employed by 26%. Moreover, three groups comprising 7% each exhibited application of proportions, incorrect statements, and simple exercises. Thirty percentage of the fourth year students among the pre-service teachers posed problems which indicated confusion with multiplication, while 25% employed through multiplication ones, and 20% posed simple exercises and 12% employed problems in the measurement category, as in the following example: "How many cakes would we have by dividing one half of the pie with one tenth of the pie?" While none of the seniors skipped the question, 7% posed problems with incorrect statements.

More than half of the first year pre-service elementary mathematics teachers skipped this question, while only one quarter of juniors did so. The sophomores who left the question unanswered were only a few. All seniors, on the other hand, provided a response. While problems with incorrect statements were most frequent with second year students, this group numbered much less in other years. The problems in the measurement category were most frequent with the second year students, while simple exercises were the most frequent choice by the fourth year students. Furthermore, the most frequently employed category for the fourth year students was confusion with multiplication, whereas sophomores did not employ this category much. The problems based on multiplication were also most commonplace with the fourth year students.

DISCUSSION, CONCLUSION, AND RECOMMENDATIONS

This study tries to analyze the problems posed with respect to multiplication and division operations with fractions, by pre-service elementary mathematics teachers in various years of training, and to shed light on how their skills and understanding in this area change. In this context, the pre-service teachers were asked to pose problems to reflect multiplication and division operations. The problems posed by the pre-service teachers were analyzed to lead to conceptual conclusions regarding multiplication and division with fractions. The study results indicate that, regardless of the year of enrollment, the pre-service teachers in the program are more successful in posing of problems regarding multiplication, compared to those regarding division. This finding is consistent with Işık's (2011) study on 4th year pre-service elementary mathematics teachers. The rates of skipping the relevant questions, failing to pose problems, is much higher with division related questions, compared to those involving multiplication. Furthermore, the first year students had the highest rates of skipping questions, whereas the fourth year students often had confusions with multiplications, or did pose problems in the form of simple exercises.

The most frequently employed category for the operation involving the multiplication of a fraction equal to one half, with a proper fraction (1/2 x 1/8), was the problems entailing the calculation of 1/8 of one half of a whole, followed by through division problems. The concept of "half" referring to the fraction 1/2 is a term from the daily life, and hence, the problems posed may contain this term. For this operation, 1st, 2nd, and 3rd year students posed mostly problems to calculate 1/8 of the half of a whole, while 4th year students preferred simple exercises.

In case of multiplication operation, simple exercises were the most frequent choice of fourth year students, while the first year students are found to have the highest skipping rates. In striking contrast, the seniors almost never left a question without an answer. The experience the pre-service teachers gained with respect to fractions, throughout the program at the faculty of education may have something to do with the tendency of the seniors to pose a problem at all times. On the other hand, lacking such experience, the first year students may also be suffering from problems with the concept of fractions. As was the case with Işık's (2011) study, pre-service teachers did, from time to time, pose problems where the result of the multiplication with proper fractions was larger than the multipliers. Yet, in the case of multiplication of two proper fractions, the result must be smaller than individual multipliers. This tendency is probably reinforced by the weakness of the conceptual understanding of fractions.

In the division of one half by a proper fraction (1/2:1/10), the confusion with multiplication was a frequent occurrence in problem posing. The first year students were again noted as the group with the highest skipping rates, while second year students most frequently employed measurement, third year students employed a similar number of instances where confusion with multiplication was prevalent, or skipped the question altogether. Seniors, on the other hand, had confusions with multiplication.

The division with fractions refers to two distinct meanings: measurement, and allocation of equal shares. However, the pre-service teachers who took part in the study had a preference for allocation of equal shares, rather than the measurement function. Furthermore, the pre-service teachers posed problems requiring division by the number in the denominator of the divisor fraction. Such findings of the study offer parallels with the conclusions of other studies in the literature (Işık, 2011; Ma, 1999).

It is evident that, regardless of the year in the program, pre-service teachers have difficulties in division and multiplication with fractions. Its reason may be related with the limited usage of fractions in daily life. Most problems observed are reflections of conceptual issues. Failure to remedy the conceptual weaknesses observed in this study might have significant repercussions on the education they will provide to their students. So that it is important to know these problems of pre-service teachers. In this context, education faculties have to take necessary measures to avoid these problems. Future studies may determine that which measures can take for these problems. Moreover, future studies may entail interviews with pre-service teachers, and investigate, in depth, the issues they face in formulating problems, and the causes which lead to more significant difficulties with divisions with fractions, compared to those faced in the context of multiplication.

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FROM CONJECTURE GENERATION BY MAINTAINING DRAGGING TO PROOF

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In this paper we propose a hypothesis about how different uses of maintaining dragging, either as a physical tool in a dynamic geometry environment or as a psychological tool for generating conjectures can influence subsequent processes of proving. Through two examples we support the hypothesis that using maintaining dragging as a physical tool may foster cognitive rupture between the conjecturing phase and the proof, while using it as a psychological tool may foster cognitive unity between them.

THEORETICAL PERSPECTIVE

Mathematics educators have been encouraging the use of technology in the classroom, and, in particular, several studies on the teaching and learning of geometry (e.g., Noss & Hoyles, 1996; Mariotti, 2006) have shown that a Dynamic Geometry Environment (DGE) can foster the learners' processes of conjecture generation and argumentation, especially in open problem situations, for which the dragging tool plays a crucial role (e.g. Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010; Leung, Baccaglini-Frank & Mariotti, 2013). In particular, research carried out by Arzarello et al. (2002) led to the description of some dragging modalities used by secondary school students when asked to solve open problems by producing conjectures in a DGE and then proving them. They describe the key moment of the process of conjecture generation as an *abduction*, related to the use of a particular form of dragging used to maintain a certain geometrical property while the figure changes as an effect of dragging one of its points. We now clarify the main theoretical notions used in the paper.

Maintaining Dragging

To shed light on the key moment described above, the first author conducted a study (Baccaglini-Frank, 2010a; Baccaglini-Frank & Mariotti, 2010, 2011), in which intentionally inducing an invariant by dragging a point was called *maintaining dragging* (an example will be shown below), and this was explicitly introduced to students between the ages of 15 and 17 in Italian high schools. The study led to a model describing cognitive processes involved in conjecture-generation when maintaining dragging is used by the solver. The findings presented in this paper stem from our interest about possible effects that use of maintaining dragging during conjecture generation might have on the subsequent proof of the conjecture, and were obtained through new analyses of the original data.

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Abduction

Peirce was the first to introduce abduction as the inference, which allows the construction of a claim starting from some data and a rule (Peirce, 1960). In mathematics education there has been renewed interest in the concept of abduction in the context of problem solving in DGEs (e.g., Arzarello et al., 2002; Antonini & Mariotti, 2010; Baccaglini-Frank, 2010b; Baccaglini-Frank & Mariotti, 2011). In this paper we will refer strictly to Piece's definition of abduction of the general form, that is: (*fact*) a fact A is observed; (*rule*) if C were true, then A would certainly be true; (*hypothesis*) so, it is reasonable to assume C is true.

Cognitive Unity

We will explore relationships between conjecture generation and mathematical proof. Studies have shown that the use of DGEs can promote conjecture generation, but not necessarily the transition to proof (e.g., Yerushalmy, Chazan & Gordon, 1993). However, interesting results have been reached on a possible continuity between these processes, leading to the elaboration of the theoretical construct of *cognitive unity*. The original term (Garuti, Boero & Lemut, 1998) was later redefined, assuming that there may or may not be continuity between the conjecturing phase and the subsequent proof produced (Pedemonte, 2007). The construct of cognitive unity has yielded great potential as a tool of analysis of the relationships between processes of conjecture generation and proofs; cognitive unity can be assessed comparing the sequences of properties logically linked during the conjecturing phase to those elaborated in the proof.

GENERATING CONJECTURES THROUGH MAINTAINING DRAGGING

We will use an example to show how maintaining dragging can be used to generate conjectures in an open problem situation. The request is the following:

Construct the quadrilateral ABCD (see Fig. 1) following these steps and make conjectures about the possible types of quadrilateral it can become describing *all the ways* you can obtain a particular type of quadrilateral. Construct: a point P and a line *r* through P, the perpendicular line to *r* through P, C on the perpendicular line, a point A symmetric to C with respect to P, a point D on the side of *r* containing A, the circle with centre C and radius CP, point B as the second intersection between the circle and the line through P and D.

The figure can be acted upon by dragging points (let us think about dragging D), and some geometrical properties can be recognized as invariants no matter how the point is dragged (e.g., "CP = PA") while others can become invariants induced through maintaining dragging (e.g., "DA = CB", "CD ∥ BA", "ABCD parallelogram"). During this kind of dragging (e.g., maintaining "ABCD parallelogram" by dragging D), new invariants can be observed as the intentionally induced invariants are visually verified (e.g., "D lies on a circle C_{AP} with centre in A and radius AP").

Recognition of new invariants during dragging can be supported by the use of the trace mark, a functionality in most DGEs. The solver may perceive the newly observed invariants as conditionally linked to the intentionally induced invariant, and express this perception in the form of a conjecture (e.g., "If D belongs to C_{AP}, then ABCD is a parallelogram").

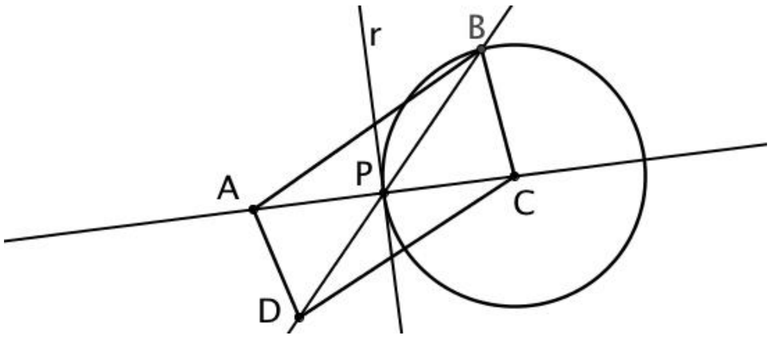

Figure 1: a possible result of the construction in the situation described above.

Our research has shown that many solvers who decide to use maintaining dragging perceive the invariants observed during dragging as *causing* the intentionally induced invariant to be visually verified, at a perceptual level, and interpret this as a *conditional link*, leading to a conjecture in the domain of Euclidean geometry (Baccaglini-Frank & Mariotti, 2010; Leung, Baccaglini-Frank & Mariotti, 2013). The process is described in further detail by Baccaglini-Frank and Mariotti (Baccaglini-Frank & Mariotti, 2011; Baccaglini-Frank, 2010a, 2010b).

MAINTAINING DRAGGING AS A PSYCHOLOGICAL TOOL

Analyses of the data from the original study have shown that students can also come to use maintaining dragging mentally, freeing it from the physical dragging support. Below is an example of how this happened (also see Baccaglini-Frank 2010a, 2010b from which these excerpts are taken).

Two 15-year-old students in the second year of an Italian high school, Francesco and Gianni, are working on the problem in the example described above. Initially, in order to obtain the desired property (that we will indicate with P_d) "ABCD parallelogram" the students have chosen diagonals intersecting at their midpoints (P_1) as the property to induce intentionally through maintaining dragging (the student holding the mouse is in bold). However their attempt fails.

Gianni: and now what are we doing? Oh yes, for the parallelogram?

Francesco: Yes [as he drags D with the trace activated] yes, we are trying to see when it remains a parallelogram.

Gianni: yes, okay the usual circle comes out.

Francesco: wait, because here…oh dear! where is it going? […] So, maybe it's not necessarily the case that D is on a circle so that ABCD is the parallelogram. Because you see, if we then do a kind of circle starting from here, like this, it's good it's good it's good it's good [he drags along a circle he imagines], and then here… see, if I go more or less along a circle that seemed good, instead it's no good…so when is it any good?

Francesco and Gianni give a geometric description of how the point D should be dragged that does not coincide with the trace mark they see on the screen as Francesco performs maintaining dragging. This leads the failure of the students' use of maintaining dragging as a physical tool, so they abandon it. Gianni, who was not dragging, conceives a condition, in his mind. This is shown in the following excerpt.

Gianni: Eh, since this is a chord, it's a chord right? We have to, it means that this has to be an equal cord of another circle, in my opinion with center in A. because I think if you do, like, a circle with center.

Francesco: A, you say…

Gianni: symmetric with respect to this one, you have to make it with center A.

[…]

Gianni: with center A and radius AP. I, I think…

Francesco: Let's move D. More or less…

Gianni: It looks right doesn't it?

Francesco: Yes.

Gianni: Maybe we found it! [Figure 2]

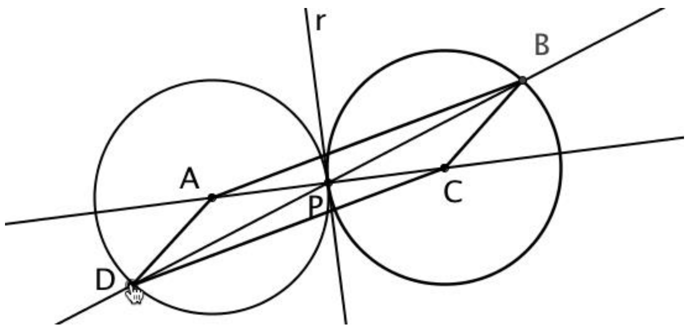

Figure 2: Francesco drags D along the newly constructed circle C_{AP}.

Gianni observes that PB is a chord of the circle C_{CP} and reasons abductively:

(*Facts*) PB is the chord of a circle, and PB = PD (P_2).

(*Rule*) If PB is the chord of a circle symmetric to the one observed then PB = PD.

(*Abductive hypothesis*) PD is a chord of the symmetric circle C_{AP} (P_3).

The abduction leads to a condition as the belonging of D to the circle C_{AP}. The students then construct the circle C_{AP} and proceed to link D to it in order to test that when D moves along C_{AP}, ABCD is a parallelogram. They seem quite satisfied and formulate the following conjecture immediately after this dragging test: "If D belongs to the circle with center in A and radius AP, then ABCD is a parallelogram".

What happened to maintaining dragging here? The students continue the exploration mentally *as if* they were dragging. Gianni seems to have interiorized the maintaining dragging tool to the point that it as become a *psychological tool* (Vygotsky, 1978, p. 52 ff.) for him (Baccaglini-Frank, 2010a, 2010b). In this case, the conjecturing process relies entirely on his theoretical control over the figure.

COMPARISON OF TWO PROOFS OF CONJECTURES GENERATED THROUGH DIFFERENT USES OF MAINTAINING DRAGGING

We now compare two proofs of conjectures generated through the two different uses of maintaining dragging described above. Our hypothesis is that when the maintaining dragging tool is physically used, there is a "theoretical gap" left between the premise and the conclusion of the conjecture, that leads to a discontinuity between the conjecturing phase and the proof; on the other hand, if maintaining dragging is used as a psychological tool, the abduction performed by the solver brings out key theoretical ingredients for the proof (similarly to what is described in Arzarello et al., 2002), fostering cognitive unity. The two proofs below are respectively by Gianni and Francesco, the students in the excerpts above, and by Ste and Giu, two 16-year old students.

Proof in the case of maintaining dragging used as a psychological tool

Below we sketch out the proof constructed by Gianni and Francesco for the conjecture: "If D belongs to the circle with centre in A and radius AP, ABCD is a parallelogram." During the conjecturing phase the following properties were used: ABCD parallelogram (P_d); diagonals that intersect at their midpoints (P_1); PD = PB (P_2); DP chord of a circle symmetric to C_{CP} with centre in A and radius AP (P_3); D $\in C_{AP}$ (P_4). During this phase the students look at the figure without dragging anything.

The proof was reached in 2 minutes, though the following steps: the circles are symmetric so AD = AP = PC = BC; the isosceles triangles APD and CPB are congruent; so PD = PB; so ABCD has diagonals that intersect at their midpoints, so it is a parallelogram.

Conjecturing phase (time proceeds from right to left)				
P_4 ←	P_3 ←	P_2 ←	P_1 ←	P_d
D $\in C_{AP}$	DP chord of C_{AP} symmetric to C_{CP}	BP = PD	diagonals intersect at midpoints	ABCD parallelogram

Proof						
P_4 and P_3 ⇒	Q_1 ⇒	Q_2 ⇒	Q_3 ⇒	P_2 ⇒	P_1 ⇒	P_d
symmetric circles and D $\in C_{AP}$	AD = AP = PC = BC	$\angle$DPA= $\angle$BPC	triangles APD CPB are cong.	BP = PD	diagonals intersect at midpoints	ABCD parallelo gram

Figure 3: Conjecture and proof generated by Francesco and Gianni.

Figure 3 shows how all the properties used in the conjecturing phase are also used in the proof: the shaded cells show properties used in both phases. In particular, it seems like the abduction generated in the conjecturing phase led to the geometrical properties – P_1, P_2, P_3 – that allowed the students to theoretically fill the gap between the premise (P_4) and the conclusion (P_d), flipping in particular the abduction between P_2 and P_3 into a deduction. A few properties – Q_1, Q_2, Q_3 – are added in the proof only in order to theoretically establish the logical relationships between the properties P_4, P_3 and P_2, used in the conjecturing phase. This is a case of cognitive unity between the conjecturing phase and the proof.

Proof in the case of maintaining dragging used physically

In order to shed light on relationships between the conjecturing phase and the proof for the second pair of students, we briefly describe how they reached a conjecture. Unlike Gianni and Francesco, Ste and Giu use maintaining dragging physically: they drag D and try to maintain the property "ABCD parallelogram" (P_d). Since they have trouble maintaining P_d, they induce a different property, that is $B \in C_{PD}$ (P_3). Doing this they eventually reach the conjecture: "ABCD is a parallelogram if PA = AD." The properties they touch on during the conjecturing phase are: a parallelogram has diagonals that intersect at their midpoints (P_1), BP = PD (P_2) through a first abduction, $B \in C_{PD}$ (P_3) through a second abduction, $D \in C_{AP}$ (P_4) using maintaining dragging, and PA = AD (P_5), another invariant observed during maintaining dragging, which they use as premise in their conjecture.

Ste and Giu conclude their proof in about 5 minutes and leave the figure static on the screen as they reason. The inferences they initially make are: CP = PA by construction (Q_1), PA = AD (P_5), $\angle CPB = \angle APD$ (part of Q_2) because vertically opposite angles. These inferences seem not to take into account properties that had been noticed during the exploration. The students are hesitant. Then Giu continues:

> Giu: So this is equal to this [he seems to point to BP and PD]…so you prove that one triangle is a 180° rotation of the other and you prove these are parallel? I don't know. […]
>
> Ste: We need to prove that B belongs to this circle here…

Giu seems to be pointing to BP and PD, stating they are equal (P_2), however, he is interrupted by Ste, and in the end this property is not used in the proof. The students try (and in the end succeed) to prove that the triangles CPB and APD are isosceles and congruent (Q_3): they claim the triangles have congruent sides of the same lengths (Q_1) and equal base angles ($\angle CBP = \angle CPB = \angle APD = \angle ADP$) ($Q_2$). Since $\angle CBP = \angle ADP$ (part of Q_2) and DA = CB (Q_4), then two opposite sides of ABCD are not only congruent but also parallel (Q_5), which proves that ABCD is a parallelogram (P_d).

Surprisingly to us, in the final proof the students never use the property BP = PD (P_2), which played a key role in the conjecturing phase, leading to a property that they found easier to maintain during dragging and that eventually led them to the property they used as a premise. Giu seems to recall the importance of the property as he points to BP and PD at the beginning of the proving phase, but there seems to be too great a "theoretical gap" between P_4 and P_3, left by the physical use of maintaining dragging, for the students to be able to theoretically bridge the gap and make use of the properties identified in the conjecturing phase. Figure 4 summarizes the properties used in the conjecturing phase and in the proof by Giu and Ste. The shaded cells show the common properties, which end up being only the premise (P_5) and the conclusion (P_d) of the conditional statement, now a proved theorem. The lack of cognitive unity between the conjecturing phase and the proof is quite evident.

Conjecturing phase (time proceeds from right to left)					
$P_5 \leftarrow P_4 \leftarrow P_3 \leftarrow P_2 \leftarrow P_1 \leftarrow P_d$					
PA = AD	$D \in C_{AP}$	$B \in C_{PD}$	BP = PD	diagonals intersect at midpoints	ABCD parallelogram
Proof					
P_5 & $Q_1 \Rightarrow Q_2 \Rightarrow Q_3 \Rightarrow Q_4$ & $Q_5 \Rightarrow P_d$					
PA = AD	CP = PA	$\angle CBP = CPB = \angle APD = \angle ADP$	triangles CPB and DPA are cong.	DA = CB and CB ‖ AD	ABCD parallelogram

Figure 4: Conjecture and proof generated by Giu and Ste

CONCLUSIONS

Although we cannot draw any general conclusions because we still have very few data on processes of proving for this kind of open problem situation, the two examples offered seem to well support our hypothesis that different uses of maintaining dragging may foster cognitive unity or rupture between the conjecturing phase and the proof of a conjecture generated by students in open problem situations. We find particularly interesting that when maintaining dragging is internalized and becomes a psychological tool, and it is no longer used physically, the abductive reasoning that takes place in the conjecturing phase seems to lead to the discovery of geometrical elements and properties that otherwise are not noticed. These can be reinvested in the proving phase, since they can be re-elaborated into the deductive steps of a proof, as in the case of Gianni and Francesco. On the other hand, in the case of physical use of maintaining dragging these geometrical elements are "absorbed" by the tool: the conjecturing phase seems to not allow the solvers to "bridge the gap" between the premise and the conclusion of their conjecture.

We believe that considerations emerging from this paper can help educators establish educational goals when geometry is taught with the support of DGEs, or at least provide them with food for thought. Important issues to further investigate are whether we want students to become proficient enough in the use of maintaining dragging for it to become a psychological tool for them. If so, we could explore how this might be accomplished in educational settings.

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IMPROVING PROSPECTIVE MATHEMATICS TEACHERS' KNOWLEDGE OF STUDENT THROUGH LESSON ANALYSIS

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The purpose of this study is to describe the role of the lesson analysis based Teaching Practice course on the development of prospective mathematics teachers (PTs) in terms of student knowledge. The study will specifically address that how does this process improve these PTs' knowledge of student? Two groups of PTs participated in the study, one that participated in a video- based course integrated under lesson analysis framework and one that attend the course offered in the conventional form. At the end of the courses, we compare two groups of PTs competence about the knowledge of the students using a video based analysis task. As a conclusion, the PTs who took part in the Teaching Practice which required video- based lesson analysis demonstrate more sophisticated levels of noticing to student ideas and teacher's feedback.

INTRODUCTION

The goal of teaching is to support learning by the student. Getting to know the student, in turn, is crucial in terms of contributing to student learning. That is why knowledge of students is among the most important elements of pedagogical content knowledge. Knowledge of student refers to being aware of students' prior knowledge of specific topics, understanding the learning difficulties and misconceptions of particular topics (Ball, Thames & Phelps, 2008; Shulman, 1986). Ball et al. (2008) emphasizes the teacher's ability to assess the level of comprehension among the students, to realize the difficulties they face, and to develop means to overcome such difficulties, for effective mathematics teaching. This statement contains a strong emphasis on the importance of focus on the comprehension, thought, and difficulties the student may have, with respect to the development of teaching competence. In a nutshell, one of the primary activities of mathematics teaching is about analysing the responses and thoughts of the students (Ball, Lubienski, & Mewborn, 2001).

Teacher training programs are designed to formally build up the teaching proficiency of prospective teachers (PTs). Even though the PTs get the chance to improve their knowledge of students through various theoretical courses, such theoretical information usually does not suffice to provide them a clear picture of the student in an actual classroom. Driel and Berry (2010) note that student's knowledge is among the most problematic issues PTs face during their teaching practices. This makes educators to seek new approaches to support the PTs' development in terms of student's knowledge. Recent approaches include a preference for educational activities focusing on how the teaching knowledge is used by the teacher, and how it supports learning by the students (Mcdonald, Kazemi & Kavanagh, 2013). One such approach proceeds with a lesson analysis and is defined as learning teaching through instruction (Hiebert,

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Morris, Berk, Jansen, 2007; Santagata & Guarino, 2011). According to Barnhart and van Es (2015), lesson analysis is about understanding and trying to interpret the student's thinking by examining the teacher-student and student-student interactions, with a view to finding out what is necessary to support such thinking. These analyses are crucial for the development of the teaching knowledge (Sun & van Es, 2015). Such practices enable PTs to notice the thoughts of the students, to reason regarding the learning by the students on the basis of the thoughts of students, and to gain experience which can be used in their in-class practices.

There are only a limited number of studies which use such analyses to support the development of PTs in terms of teaching knowledge (Barnhart & van Es, 2015; Santagata & Yeh, 2014). This study aims to reveal how the Teaching Practice course, which offers a chance to have a video-based analysis of and reflection on learning-teaching activities, affect the development of PTs' knowledge of the student. In case the study implemented in Turkey, which is characterized by a distinctive system of education, leads to positive results, it will help provide new experimental evidence regarding the efficiency of the lesson analysis. Furthermore, the study investigate the development of the PTs' skills to lesson analysis, within knowledge of student. The study will specifically address the following research question. How does this process improve these prospective teachers' knowledge of student? Therefore, the presentation is effectively a part of a more comprehensive project, and will offer product-focused insight into the change and development in PTs.

THEORETICAL FRAMEWORK

This study is based on noticing and lesson analysis theoretical frameworks. According to the noticing, three fundamental indicators of the teacher's awareness of classroom interactions are identifying what is crucial for learning-teaching activities in the classroom, establishing the link between the classroom interactions and their associated general learning-teaching principles, and finally using existing knowledge to reach inferences regarding the nature of classroom interactions (Jacobs, Lamb & Philipp, 2010; van Es & Sherin, 2002). The lesson analysis, on the other hand, is a systematical perspective focusing on the teaching activities by the teacher and the learning by the student. In this context, an assessment of the effectiveness of teaching through lesson analysis requires the skills to determine the thoughts of the student, and to develop her interpretation skills (Sun & van ES, 2015). Furthermore, systematical analyses such as those provided by lesson analysis require noticing-related skills, and therefore, help develop noticing skills (Barnhart & van Es, 2015).

METHOD

This study was carried out "Teaching Practice" course offered to 4th year students in the secondary mathematics teaching program. The first group (NLTP) comprising 12 PTs the course offered in the conventional form, whereas the second group (LTP) comprising 12 PTs who volunteered took the course integrated under the lesson analysis framework. The first author personally offered the Teaching Practice course for both groups.

Lesson Analysis Supported Teaching Practice (LTP)

PTs were separated into 4 groups of 3. Each group spent 6 hours per week at the school. The whole procedure took 10 weeks. During the first three weeks the PTs were asked to observe the assigned teachers and prepare a reflection report taking into account the main theme in Table 1. In the remaining 7 weeks, they were expected to engage in lesson analysis on video regarding their own teaching practices.

Theme	Remarks
Identification	Identification of the cases where students have difficulty in learning and make mistakes.
Interpretation and statement of the reason.	Clarifying cases inhibiting comprehension of and misleading students.
Providing recommendations.	Coming up with recommendations to eliminate learning difficulties and mistakes by the students in case of repeating a topic.

Table 1: Lesson analysis framework

Throughout this process, the first researcher made video recordings of 4 hours of teaching practice by each PT in an actual classroom. Each video was provided to PTs on the same day, with the expectation that they would engage in lesson analysis and present a report. During routine weekly meetings at the university, a pre-selected video recorded during the week was discussed with PTs, with particular emphasis on the difficult points from the perspective of the student, reasons thereof, and proposed solutions.

Teaching Practice without Lesson Analysis (NLTP)

Formally speaking, the 10 weeks implementation period with this group is similar to that of the other group. Each PT had 4 hours of teaching practice during the semester. Throughout the semester, PTs were required to prepare reports to reflect the general structure of the lesson they and their friends had given. The routine weekly meetings at the university. Each PT was observed personally by the first author, for at least one hour of class, and was provided feedback.

Data gathering tools

The reports drawn up by PTs within the framework of the course, field notes by the researcher, and the records of the meetings are the major data sources of the study. Furthermore, in the end of the procedure, PTs in both groups were asked to provide an analysis regarding a 5 minutes video clip taken in an actual classroom. The PTs were asked to take notes of what they notice about the interaction at the classroom while watching the video, and to provide details of such notes once the video was over. The video is from an actual classroom where the ability to '*Multiply a natural number with an algebraic expression'* is taught. The video clip shows the teacher failing to provide appropriate feedback in response to a mistake by the student. The PTs are expected to notice the teacher's behaviour, and develop recommendations on how to behave.

Data analysis

The reports drawn up by PTs about the video task were read completely by the first researcher. In this context, the researcher focused on what the PTs noticed, how they interpreted these, and if they presented any recommendations or not, and then categorized the statements of the PTs in terms of their similarities and differences. The categorization was based on the framework for learning to notice students' mathematical thinking, developed by van Es (2002). When marking the levels of the PTs' statements, identification and interpretation of the teacher's behaviour regarding the mistake served as the key. The final form of the rubric (see Table 2) was used to review once again the reports by the PTs.

Levels	Definition
Level 1	Just the description of the teacher's behaviours. Failure to notice the mistake of the student and the teacher's intervention regarding the mistake.
Level 2	Noticing the student's mistake. Settling with just the description of the teacher's behavior regarding the mistake.
Level 3	Noticing the student's mistake. Limited interpretation about the nature of the teacher's intervention regarding the mistake. The interpretation containing relatively superficial statements which are not related to the students, such as *'the teacher's behaviour is very acceptable or inacceptable*'.
Level 4	Noticing the student's mistake. Interpretation about the nature of the teacher's intervention regarding the mistake. The interpretation containing statements focusing on the student, such as *'the teacher disregarded the student's error, and failed to elaborate on why the student made the mistake'*, and furthermore containing recommendations on how to intervene on such mistakes.

Table 2: PTs' lesson analysis rubric concerning the teacher's behaviour

FINDINGS

The analysis results for both groups are shown in Table 3.

Group	Level 1	Level 2	Level 3	Level 4
NLTP	Betül, Emel, Nur, Derya, Gürcan, Mine, Hamit, Ezgi, Ümit, Sena	Erhan	Funda	
LTP	Mustafa, Tülay Ayşe, Sibel	Nalan	Rana	Dilek, Nursel, Nazan, Elif, Bahar, Hatice

Table 3: Lesson analysis levels of the prospective teachers in both groups

A glance at Table 3 suggests that the PTs in the LTP performed much better in the video analysis, in comparison to those in the NLTP. 10 PTs in the NLTP ranked in level 1, whereas just 4 PTs in the LTP did so. This fact indicates that the majority of the PTs in the NLTP failed to take note of the student's mistake and the teacher's behavior in response. On the other hand, 8 PTs in the LTP noticed the teacher's behavior in response to the student's mistake, while only 2 of the PTs in the NLTP did so. None of the PTs in the NLTP ranked in level 4, whereas 6 PTs in the LTP ranked at that level. This effectively means that 6 PTs not only noticed the student's mistake and the teacher's response, but also commented on the acceptability of the teacher's behavior, as well as made recommendations on how she should have.

6 students in the LTP ranked in level 4. These PTs noted in their reports the student's mistake and the teacher's reaction to the mistake and criticized the teacher for failing to provide feedback for the student's mistake, and to investigate the cause of the mistake. A section of the report by Bahar is quoted below.

> A student went to the blackboard. She proposed a different solution. She said that the first row of the algebraic tile represented x+6, while the column represented 3x. No statement was made on why the student committed the mistake. One could have asked the student “Why do you think so?”, “Do you get the same result if you go through this route?” etc. Yet, the teacher proceeded directly with the explanation. I believe the students had misconceptions regarding operations using algebraic tiles. Perhaps one could have discussed their reasons. The student who committed the mistake should have been questioned about what she was thinking.

Bahar noticed that the teacher disregarded the student’s mistake. Her statements emphasized the need on part of the teacher to understand the student’s mistake, as well as to ask questions to make her realize the mistake as well.

None of the PTs in the NLTP provided statements which is on par with level 4. Level 2 and 3 had one PT from each group. Both levels include the PTs who noticed the student’s mistake. Level 3, in contrast to Level 2, however, requires the PTS to provide statements regarding the nature of the feedback the teacher provided in response to the mistake. In this context, just two PTs in the NLTP can be named to have noticed the student’s mistake and the teacher's behavior in response. An excerpt of the response by Funda, who ranked in level 3, is provided below.

> The student who provided the first response to the teacher's question about the algebraic expression for this model said "there are 3 x's; that is 3x; and as there are 6 1s, we have 3x + 6". The second student also wrote 3x + 6. The students had difficulty in noticing the second way. When the students experienced difficulty, the teacher asked them how they could achieve this by using multiplication. The mistake of a girl who provided an incorrect answer was explained by the teacher, who made sure that the mistake was corrected.

In her notes, Funda stated that a girl made an mistake, but did not specify where and how she did so. Actually, here the teacher's statement is not related directly with the student's mistake, but instead is a general remark towards the whole class. The response by Funda, as it notices the student's error and the teacher's response, but provides only a limited and superficial interpretation of the teacher's behaviour.

4 of the PTs in the LTP and 9 of the PTs in the NLTP ranked level 1 with their notes. The notes by Hamit(NLTP) are quoted below.

> The teacher asked for the algebraic expression for the model she formulated. Students reached the answer 3x + 6. Then, the teacher asked "Can we express it in another way?". One of the students gave the correct answer. She wrote it on the blackboard. (3. (x + 2)). The teacher, in turn, explained the left side as 3, and the upper side as x + 2.

A glance at Hamit's remarks indicates that he tries to summarize what is going on in the classroom. He did not delve on the mistakes of the students and the reaction of the teacher at all. The notes by Mustafa (LTP) in level 1 are quoted below.

> The teacher developed the model and asked the students to express in algebraic terms. Another student wrote 3x on the first column, and x+6 on the first row. Then the teacher wrote (x+2) on the first row, and asked "how many counts of this row do we have?" before proceeding to write (x+2).3 and explaining it. Here, the teacher's efforts to have the problem solved by the students, and her refrain from telling the result outright, did help. This helped increase the students' participation in the class even more; they tried to develop new ways, and the class became more enjoyable. However, an mistake in the use of algebraic tiles for explaining this gain made it difficult for the students to find the answer and establish a relationship.

Mustafa's remarks reveal that he provided a general description of the events in the classroom, but failed to focus on the student’s mistake. Only in his last line mention the meaning of the algebraic tiles being difficult to grasp for the students, due to a lack of clarity. Mustafa's response matches level 1 better, as all it contains a description of the behaviors of the teacher and the students. However, one can also say that Mustafa's statements are more qualified compared to the statement provided as an example of level 1 statements by the NLTP group.

CONCLUSIONS

The purpose of this study is to analyse the development of the lesson analysis skills by PTs, and to state the development's contribution to the knowledge of students. The PTs who took part in the "Teaching Practice" which required video-based lesson analysis had their skills to analyse a lesson had improved in terms of identifying learning difficulties. The video clip shown to PTs contained general remarks to the classroom, without the teacher providing any feedback to the mistake by the student. This was noticed by the majority of PTs in the LTP, while the NLTP did not realize it at all. This finding is consistent with Santagata and Guarino (2011) conclusion that video-based activities help PTs in getting the details of the students' thinking and identifying the teaching acts of the teacher to render such thinking visible.

6 of the PTs in the LTP achieved level 4 in noticing the teacher's behaviour, whereas the PTs in the NLTP did not so. The PTs who ranked in this level in the LTP not only noticed the student’s mistake and the teacher's behaviour concerning the mistake, but also interpreted the teacher's behaviour in terms of its effectiveness and provided recommendations on what to do. Usually the PTs proposed for the teacher an investigation of the student's mistake, and interaction with the student to make the reason of mistake known. This suggest that the PTs in the LTP have the awareness of the need to focus on the students' thinking in the teaching process. The PTs in the NTLP, on the other hand, chose to confide themselves to a description of the classroom interaction shown in the video; just two of them noticed the student's mistake in the video. However, they fell short of focusing on the cause. This suggests that the PTs in the NLTP lack the awareness of the need to inquire about the student's mistake. As the teachers' awareness of the importance of the students' thoughts increase, their classroom practices will improve as well. Indeed, according to Levin, Hammer & Coffey (2009), PTs developing the awareness of the need to grasp the students' thinking will lead them to practices to investigate the students' thinking. The most important problem faced by novice teachers and PTs is teaching in a perspective isolated from the students (Barnhart & van Es, 2015)**.** This is especially the case with the analyses provided by the PTs in the NLTP. Their analyses usually provided only a description of what the teacher did, and lacked a focus on what the students did. In conclusion, the PTs in the LTP focus, arguably, more on the difficulties faced and mistakes committed by the students.

Some PTs in the NLTP failed to provide level 4 analyses. However, a glance at the analyses by such PTs indicates that even their analyses contain more comprehensive remarks. These PTs noticed the difficulty the students had, while failing to provide an interpretation of the teacher's behaviour in response, indicating a lack of practice to achieve such change. This can perhaps be achieved through a longer course.

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CONTEXTUAL-BASED SIMILARITY TASKS IN TEXTBOOKS FROM BRAZIL AND THE UNITED STATES

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Textbooks from Brazil and the United States were analyzed with a focus on similarity and contextual-based tasks. Students' opportunities to learn similarity were examined by considering whether students were provided contextual-based tasks of high cognitive demands that included missing or superfluous information. Results from five textbooks are provided and, examples of context-based tasks are discussed.

INTRODUCTION

This paper reports on a portion of a project involving two countries, Brazil and the USA. Our goal is to analyze the presentation of geometry in textbooks from both countries. For this paper, we focus on the concept of similarity and the use of contextual problems. Similarity often provides opportunities for students to investigate real-world problems. Even though each country throughout the world has different textbooks, the results of our work can contribute to existing knowledge about this most used resource in the classroom (Valverde et al., 2002; Haggarty & Pepin, 2002).

Many researchers have examined textbooks; this line of research is not new. Fan (2013) identified several aspects of textbooks that could be examined, and several researchers have investigated mathematics content in textbooks. For example, Haggarty & Pepin (2002), Schmidt et al. (1997) and Wijaya et al. (2015) discussed opportunities students have to learn particular content and focusing on the topic of angles. A deep analysis of the possibilities of textbook tasks to supFport students' knowledge was conducted. We focus on the opportunity to learn and will present results from the analysis of the ways in which three textbooks from Brazil and two textbooks from the USA present the concept of similarity and use contextual problems.

THEORETICAL FRAMEWORK

The idea of opportunity to learn (OTL) is well established. Carroll (1963) described it as the amount of time students spend learning particular topics. Husén (1967) defined it as "one of the factors which may influence scores [...] whether or not the students have had an opportunity to study a particular topic or learn how to solve a particular type of problem" (p. 162). Since this period, several researchers have used OTL to analyze students' learning experiences and learning resources such as textbooks (e.g., Floden, 2002; Haggarty & Pepin, 2002; McDonnell, 1995; Tornroos, 2005). McDonnell (1995) contextualized OTL as an education indicator, describing research

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 59–66. Szeged, Hungary: PME.

about curriculum, school and classes. For her, OTL is used to describe the complexity of the schooling process and "although designed as a technical concept to ensure valid cross-national comparisons, OTL has changed how researchers, educators, and policymakers think about the determinants of student learning" (p.305). Even its applicability as a policy instrument is limited.

We adopted the description of OTL provided by Wijaya et. al (2015) that considers

> three aspects of OTL [that] are crucial to develop the competence of solving context-based tasks. The first aspect is giving students experience to work on tasks with real-world contexts and implicit mathematical procedures. The second aspect is giving students tasks with missing or superfluous information. The last aspect is offering students experience to work on tasks with high cognitive demands. (p.46)

They also describe contextual problems as the types of problems that present a situation referring to the real world or a scenario that can be imagined by students. These contexts and scenarios may include personal, scientific, occupational, or public information. However, just because a problem involves context does not mean that the problem will be of high cognitive demand. Thus, it was also important to consider the type of thinking that was required of students to solve these context-based problems. As discussed in Boston and Smith (2009), different types of tasks require different kinds of thinking that can influence the opportunities students have to learn particular mathematical ideas.

CONTEXT AND METHOD

For this project, five textbooks were selected for analysis: three textbooks from Brazil and two textbooks from the USA. The textbooks from Brazil presented mathematical ideas in an integrated manner. That is, similarity was included in a textbook that also included topics such as algebra and statistics. At the high school level, textbooks in the USA tend to present mathematics in two different ways. Some textbooks have a single subject focus. For example, in 9th, 10th, and 11th grade students might use Algebra I, Geometry, and Algebra II textbooks. Similarity would be included in the Geometry textbook and the main focus of the text would be on geometry topics. Other textbooks in the USA present topics in an integrated manner, similar to Brazil and other international countries. Students in 9th, 10th, and 11th grade might use textbooks that are titled Integrated Mathematics I, Integrated Mathematics II, and Integrated Mathematics III. We selected textbooks from the USA that presented mathematical topics in an integrated manner. In these textbooks, similarity was included in an Integrated Mathematics II textbook. Students would typically use this textbook during the 9th or 10th grade when they are 14 or 15 years old. In Brazil, similarity is presented in the 9th grade (14 years old).

We started identifying the physical characteristics of the books (e.g., location, number of pages on similarity, similarity topics addressed and sequence of presentation, structure of a lesson, number of contextual tasks and their cognitive demands). After that, we analyzed the OTL on the contextual-based tasks (approach of contextual-based tasks, real experiences, superfluous/missing information, cognitive demand).

All the tasks about similarity were coded by the first author to identify the low or high level of cognitive demand. Afterwards, the reliability of the coding was checked through an additional coding by the second author who coded a random selection of about 41% of the tasks. After some adjustment, 92% agreement was reached.

After that, we focused on contextually-based tasks. In Figure 1, we show an example of this type of task.

Refer to the map at the right. 3rd Avenue and 5th Avenue are parallel. If the distance from 3rd Avenue to City Mall along State Street is 3201 feet, find the distance between 5th Avenue and City Mall along Union Street. Round to the nearest tenth.

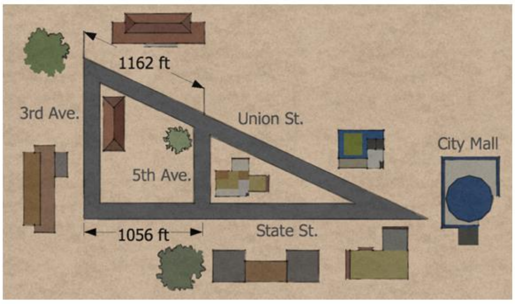

Figure 1

Our interest was in the contextual tasks like the one above. When considering a problem such as this, we reflect on the three components of OTL. In particular we want to know whether the context is real or can be imagined, whether there is missing or superfluous information, and to determine the cognitive demand of the task.

RESULTS

The physical characteristics of the textbooks

When analyzing the textbooks, we began by considering the physical features of the book. This included the total number of pages and the number of pages focused on similarity (See Table 1). We noticed that even though there were more pages devoted to similarity in some of the books, the percentage of pages focused on similarity ranged between 7.7 and 10.4%.

Table 1

	Total number of pages in the textbook	Number of pages focused on similarity	Percentage of the book focused on similarity
Book 1-BR	272	21	7,7%
Book 2-BR	270	28	10,3%
Book 3-BR	240	25	10,4%
Book 1-USA	1322	104	7.8%
Book 2-USA	941	76	8,1%

To look more closely at the contextual tasks focused on similarity, we determined whether the tasks were of high or low cognitive demand (See Table 2).

Table 2

	Total of tasks[i]	Total of contextual tasks	Low level in contextual tasks	High level in contextual tasks
Book 1-BR	80	11 (13,8%)	5 (45,5%)	6 (54,5%)
Book 2-BR	140	12 (8,6%)	6 (50%)	6 (50%)
Book 3-BR	70	20 (28,6%)	9 (45%)	11 (55%)
Book 1-USA	360	58 (16,11%)	41 (70,7%)	17 (29,3%)
Book 2-USA	538	79 (14,7%)	64 (81%)	25 (19%)

Looking at table 2, we note that having more tasks does not suggest more high level tasks (proportionally). More tasks, however, offer options to the teacher, who can choose which tasks he or she wants to explore with students.

Contextual similarity tasks

Considering the OTL, we analyzed the contextual-based similarity tasks based on the three mentioned aspects (Wijaya et al., 2015). First is the opportunity to work on **tasks involving the real-world**. As Dietiker & Brakonieck (2014) suggest, one must consider, **what is the real-world?** Do the tasks require students to consider important aspects of reality? Do the tasks require students to take a critical perspective and use information about the real world that they know?

To look at the contextual-based task focusing on the real world, we considered three types of problems: 1) problems most students can relate to and make sense of; 2) problems students with particular experiences can relate to; and 3) problems that have contexts that students are not likely to encounter in their everyday lives.

A type 1 task is shown in Figure 1. Most students have experiences with maps and addresses. We can indicate another group of contextual tasks (type 2) for which the context may not be relevant to students. Tasks that focus on rural or urban issues, for example, can be relevant to one group of students but not to another. It is important for the teacher to consider to what extent the context of this problem is relevant to the students in the class.

Certain medical treatments involve laser beams that contact and penetrate the skin, forming similar triangles. Refer to the diagram at the right. How far apart should the laser sources be placed to ensure that the areas treated by each source do not overlap?

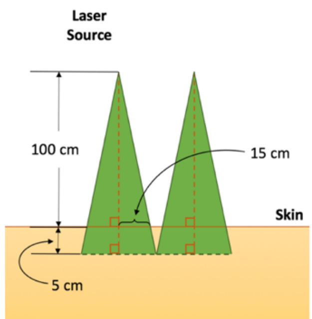

Figure 2

On the other hand, we found some tasks (type 3) about which we could ask: who has this reality? Or, is it possible to consider it as a reality? Consider the task of Figure 3[ii]

[i] We considered a task with three items as three tasks.

[ii] The tasks from Brazilian textbooks were translated to English.

that depicts a flying saucer. We do not find any problem with the mathematical question. **But is this a good example of context-based task?** When our students solve this kind of problem, are they critical about the context?

A person said he saw a flying saucer. To mark the disc position, he placed a broomstick on the floor, pointing to the flying saucer. He saw that the stick formed a horizontal angle of 30°. Another person said he saw the same object at the same time. But the flying saucer was straight over his head, vertically. The distance between the two persons was 600 meters.

Using a square, draw a triangle with a 30° angle and another at 90°. Note that it will be similar to that shown in the figure. Now, believing what the two people said, and using the measurements of the design, calculate the height the saucer was at.

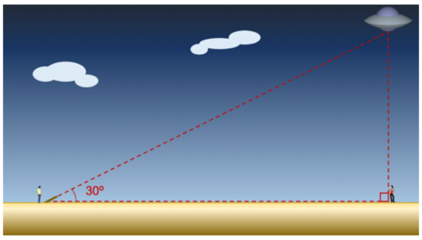

Figure 3

We cannot change the textbook's tasks. Teachers typically skip this type of problem. Our reflection on this paper is to think about how to use this kind of task as an opportunity for students to be critical. This is contextual and educative, teaching students not to accept everything without first thinking about it. Based on Dietiker & Brakonieck (2014), we can help students develop sophisticated ways to interpret problems and think geometrically if we encourage them to approach problems critically.

Another example of this type task is an example of a flag shown in Figure 4. Students are asked to determine if the triangles in the flag are similar. We wonder, **is this a good example of a context task?** Why it is important to know if the triangles are similar? Is this a real problem?

A scuba flag is used to indicate there is a diver below. In North America, scuba flags are red with a white stripe from the upper left corner to the lower right corner. Justify the triangles formed on the scuba flag are similar triangles.

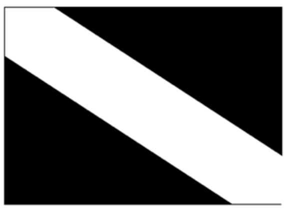

Figure 4

The second aspect of OTL is considering tasks with missing or superfluous information. As we discussed in the theoretical framework, when we talk about superfluous information, we usually link this idea with problems that have more data than we need. This is important so students learn to choose the information that is really important to the problem. In a real life, we have to decide among all information available which pieces are important to solve the problem.

Here we are reflecting about something more. Again we propose some questions to be explored with students: **What is considered superfluous**? What kind of information do we not need?

On the task shown in Figure 5 we could say the same about the information to solve the problem: the solution expected uses similarity, the measurements are sufficient to calculate the height of the building, etc.

To measure the height of a building, Mary did the following: tied a wire on top of the building; then set the other end of the wire on the ground 5 meters away from the base of the building.
Then, at a height of 5 meters from the ground, tied another wire, parallel to the first, fixing it in the ground, 2 meters away from the base of the building. Draw this situation and determine the building height.

Figure 5

But why would Mary tie a wire on top of the building and the other end five meters away from the building? If she can have a wire in the top, why she doesn't keep the wire close to the building and measure the distance from the top to the foot of the building? We are inviting reflection here about the concept of superfluity. What is not necessary in this task is not the extra information. The math solution proposed is extra work to do in a real situation. It is important for students to critically analyze that the best solution in the real world is not always the solution of the textbook.

The last aspect of OTL is "offering students experience to work on tasks with high cognitive demands" (Wijaya et al., 2015, p.46). When we read this, we think about the importance of offering students tasks that make them investigate, explore, think, and not simply repeat procedures. However, it is important that these tasks relate to what students are learning in class. The student needs to analyze the situation, see that it has more than one solution, etc. But **is this aligned with what students are learning in class**?

Edward and Lia were furnishing a room with two arrangements. To do this, they chose the following furniture: a sofa that measures 2m long and 1m meter wide, an armchair measuring 1m long and 1m wide, and a square base rack that measures 75 cm per side. For the dining room they chose a table that measures 1.5m long by 75cm wide and six chairs.

Each chair occupies a square area of 0.25 m^2.

Look at the floor plan for Eduard and Lia's room.

a) will the couple's room hold this furniture?

b) what would be the way to arrange them?

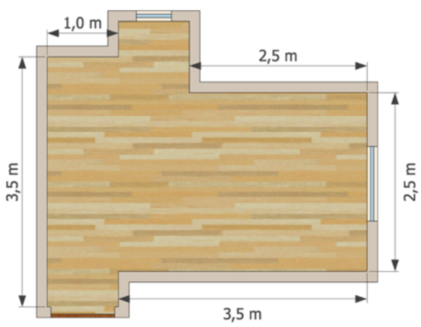

Figure 6

How is similarity explored in this task? The relation with similarity is not explicit in the task. When the student is solving this problem, is he asking why it is proposed to learn similarity? Is the teacher exploring this?

DISCUSSION

We want to explore further the previous issues related to context presented in the previous examples. The principal function of contextual-based tasks is to prepare the students to solve real problems. What we could note in evaluating textbooks of Brazil and the USA is that most textbooks have few contextual problems. More than this, we can reflect about how students can become better solvers with problems that do not make students think about real-world problems (like flying saucer). Do we prepare them to be critical with the situation they are given? Based on Dietiker & Brakonieck (2014), it is reasonable to assume that critical discussion/reflection could help students develop sophisticated ways to interpret problems and think about geometry, becoming good solvers in real situations.

As we said, we cannot change what is presented in the textbooks, but we can do more than just find the answer or skip the task; we can use this type of task to discuss critical aspects of contextual tasks with our students and assure the cognitive demand of the task is high. We can not only wait for a high cognitive-demand task given by the textbook to make the task a high-level task. As shown in the tables, the textbooks have few contextual problems of this level. As teachers, we can explore the tasks with critical questions and change them to be high, investigating the description of reality in the tasks.

Most authors mentioned in this paper and many others writing about OTL theory classify tasks as high and low, but we believe that is necessary to do more than that. It is a good process, and to identify high level of cognitive demand is important to teach with tasks that contribute a lot with the learning, but as we show, it is not enough. Some of the tasks make the students think and reflect, but maybe not in the context of the study.

Acknowledgements

This research has been financed by FAPESP (Process 2013/22975-3) and by CAPES (Process BEX 0612/15-4).

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THE DRAGGING GESTURE – FROM ACTING TO CONCEPTUALIZING

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Several studies emphasize students' difficulties in dealing with decimal numbers. Taking up this problem, we examine the potential of a digital place value chart to constitute decimals as a structure in a design-based research study. In this paper, we focus on results of the analysis of how "dragging-movements" representing bundling and de-bundling in this digital place value chart supported the epistemic process of grade 5 students.

INTRODUCTION

In learning mathematics a core role is attributed to the use of representations that mediate the handling of mathematical objects (Duval, 2000, 61), as these are not directly perceivable and manageable. Especially for young children, mathematical objects should be represented by concrete material that allows concrete activities to get access to the underlying mathematical idea (cf. Söbbeke, 2005, quoted in Scherer & Moser Opitz, 2010, 75). To construct mathematical structures represented in an activity, it is likewise important to afterwards detach from the concrete material. Following recent research, gestures have "the potential to serve as a unique bridge between action and abstract thought" (Goldin-Meadow & Beilock, 2010, 664), because they "emerge from the perceptual and motor simulations that underlie embodied language and mental imagery" (Hostetter & Alibali, 2008, 502). However, little is known about how gestures transform concrete actions into epistemic processes. The project DeciPlace (Behrens, in press) addresses this question by investigating how dragging-actions on a digital place value chart on the iPad (designed by Ladel & Kortenkamp, 2013) may support students' conceptualizing of decimal numbers. The topic of decimal numbers is used as an example because of its relevance and the difficulties students have with the underlying concept (cf. Behrens (in press); Steinle & Stacey, 2004),

THEORETICAL FRAMEWORK

The project is conducted as a design-based research project in which tasks are developed to support learners in the extension of the decimal place value system from natural numbers to decimal numbers. This is done by means of a digital place value chart on the iPad. In contrast to traditional place value charts, our digital one keeps the decimal number invariant when bundling and de-bundling takes place. This invariance is meant when we talk about the structure of a decimal number. In the chart, bundling and de-bundling are executed by dragging tokens from one field to another (see figure 1), hence, dragging-movements play a crucial role in the process of acquiring the

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 67–74. Szeged, Hungary: PME.

structure of decimals with the aid of the digital place value chart. These dragging-movements may be observed as concrete actions on the one hand and as gestures on the other.

Krause (in press) has explored gestures learners used while solving mathematical problems. She identified that gestures can comprise various epistemic functions related to language and other semiotic resources. Her study only explores spontaneous gestures, it does not show how specific actions such as dragging-actions can be initiated and transformed into gestures to support the learning of a specific concept. However, Krause's results can serve as a theoretical frame to investigate gestures which emerge from „dragging-actions“ initiated by the use of the digital place value chart on the iPad.

Following Krause (in press) gestures contribute to perform and form epistemic actions in collective epistemic processes. To examine how the students collectively acquire the decimal number as a structure the epistemic process can be described by an epistemic action model, the GCSt-model, shaped by three epistemic actions (Bikner-Ahsbahs, 2005): At the beginning of an epistemic process actions of *gathering* small and meaningful elements like examples, counter-examples, associations, results, patterns or elementary statements are performed. This prepares the consecutive epistemic action of c*onnecting* mathematical meanings by recognizing relations, by reasoning or by summarizing results. By recognizing regularities or exemplary results these connections can pass into *seeing a structure*. (Bikner-Ahsbahs, 2005, 202).

Krause (in press) has used this GCSt-model to describe epistemic functions of gestures as ways for nurturing epistemic actions. As gestures we characterize all spontaneous movements of hands and arms that accompany speech without functionally operating on something (McNeill, 1992, 37; Kendon, 2004, 15). Because of their ambiguity, gestures are interpreted in relation to speech and precedent actions that indicate the context. Krause has shown that these gestures may "refer to the representation of an object on [three] different levels" (Krause, in press, 138):

- On the first level considered as the *level of the concrete* "gesture refers to something actually represented in a fixed diagram" and thereby "works as an index to hint at something already represented" (ibid., 138).
- On the *level of the potential* (level 2) "gesture is embedded in a fixed representation but does not merely refer to an already fixed concrete component" and "represents a 'hypothetical something' not there but potentially 'thought into' a present diagram" (ibid., 138).
- Gestures on the third and *free level of reference* are performed in the gesture space "without being dependent on a present referential frame", so that "the interpretation of the gesture is detached from the concrete" (ibid., 138). Following Krause "these gestures may reveal a more conceptual than contextual idea of a mathematical situation or object." (ibid., 138).

Through these three referential levels, Krause (in press) is able to describe how specific gestures develop and detach from the concrete towards a use independent from concrete recourses such as inscriptions. Based on this theoretical background we can now be more precise in formulating the research questions to be answered in this paper:

- With respect to the referential levels, how do dragging-gestures emerge from dragging-actions initiated by the digital place value chart?
- With respect to epistemic processes, how can dragging-gestures support conceptualizing the decimal number as a structure?

METHODOLOGICAL CONSIDERATIONS

Data collection:

These two research questions are investigated through teaching experiments with ten pairs of students (grade 5 & 6, age 10-12) being conducted in a cyclic process of design and analysis. Each student pair participated in two consecutive teaching experiments guided by the researcher. These teaching experiments lasted about 90 minutes each and were videotaped from three perspectives (frontal, lateral and from above) to capture all verbal utterances, actions, gestures, etc. The video data has been completely transcribed including verbal utterances and non-verbal activities as well as gestures.

The digital place value chart:

The specific tasks of the teaching experiments are particularly constructed for the use of the digital place value chart on the iPad. To create the representation of a number tokens can be inserted directly by tapping in the respective column of the chart. The fundamental characteristic of the digital place value chart concerns the possibility to automatically de-bundle (see figure 1) or bundle (if possible) within the chart by dragging tokens from one column to another, while keeping the represented number invariant. If bundling is not possible, the token slides back to its initial position. The represented number can be additionally displayed in standard notation above the chart (for a detailed description of the digital place value chart see Behrens (in press)).

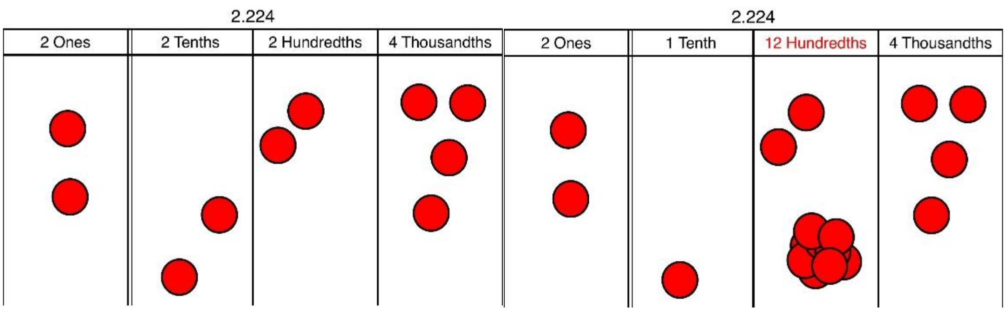

Figure 1: De-bundling a token from tenths to hundredths in the digital place value chart

The tasks:

Exploiting the automatic bundling and de-bundling within the chart, the first teaching experiment focuses on the structure of the decimal place value system. The task is to find as many representations of specific numbers as possible through bundling and de-bundling. The extension of the place value chart of natural numbers by tenths and hundredths is motivated by extending the quantity of representations, recognizing that particularly the base-ten property of the place value chart's structure (cf. Ross, 1989) remains preserved. In the second teaching experiment the completely bundled representation in the place value chart is used to prepare the introduction of the standard notation of decimal numbers. In this case, the digital place value chart can support bundling activities by enabling trials and systematic tests concerning when and how bundling is possible. Furthermore, through the possibility to display the represented number the students have the chance to explore the relation between the representation in the place value chart and the adequate standard notation.

Methods of data analysis:

In this paper we focus on the analysis of the epistemic process of the first student-pair we examined. This analysis has been conducted in three steps:

(1) In order to address dragging-movements, the data was first scoured for scenes of "dragging" either as an action on the iPad or as a hand moving gesture to the left or to the right distinguishing the three levels of reference (Krause, in press). Dragging as a gesture is identified when there is a horizontal movement to the right or to the left without lifting and declining the hand, mostly this was found to be accompanied by verbal utterances referring to "dragging". Dragging-related gestures as pointing successively on two columns within the place value chart have been coded separately. Regarding the referential level of gesture we have to adapt this concept on the application of an iPad with a touchpad-surface such that gesturing on the concrete level of reference can also mean performing operations on the iPad.

(2) In a second step, all scenes of "dragging" were analyzed by describing the epistemic process through the epistemic actions gathering, connecting and structure seeing (cf. GCSt-model, Bikner-Ahsbahs, 2005).

(3) Finally, these two analyses of the "dragging-scenes" were matched to examine how dragging on the three referential levels supports building the structure of decimals.

DATA ANALYSIS & RESULTS

By analyzing the dragging movements within the epistemic process of the two students we detected three different modes of dragging: the (1) practical, (2) operational and (3) structural dragging.

Practical dragging appears particularly when the digital place value chart is introduced and the students have not yet discovered how it works, so that they just drag tokens within the chart or utilize it without scrutinizing. Therefore, practical dragging is completely dependent on the concrete digital place value chart and it is performed on the concrete referential level. Within the epistemic process it can particularly be observed in phases of gathering.

In the first teaching experiment, the students were asked to find as many representations of a specific number in the place value chart as possible. By dragging tokens from hundreds to tens the students utilized the digital place value chart on the iPad to generate different representations of the number “101”.

1 Bella: I’ll just try (*drags a token from hundreds to tens within the digital place value chart on the iPad (in the tens’ column ten tokens emerge), see figure 2*) Woah

2 Hanna: Ten and One.

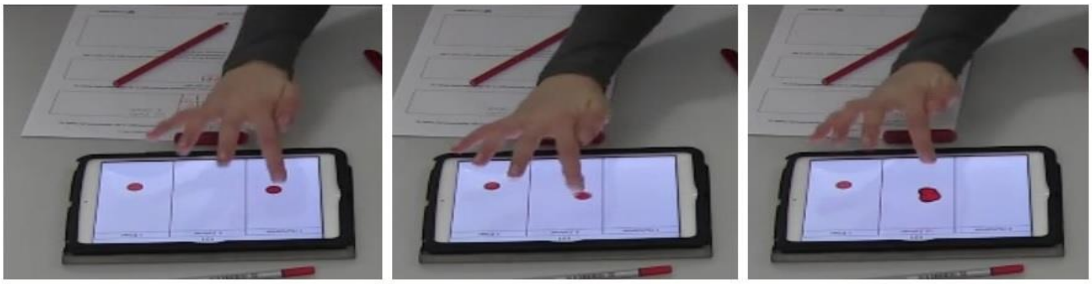

Figure 2: De-bundling as dragging from hundreds to tens

Interpreting the verbal utterance of Bella and her reaction of surprise and astonishment (line 1) on the de-bundling from hundreds to tens we can assume that she does not anticipate what will happen when dragging tokens from hundreds to tens, and especially not why this happens. Hanna’s denomination of the result (line 2) confirms our assumption and even indicates that at that time there is no need for the students to bring this observation into question, but to just write the result down on the worksheet.

When the students have figured out how bundling and de-bundling within the digital place value chart works, they are able to foresee the outcome, so that they can use both transformations through dragging on purpose. That is what we call *operational dragging*. It requires and indicates phases of connecting that facilitates the accomplishment of dragging with the intended goal.

In the second teaching experiment the students are asked to match different representations with the same value. The two students we examined pursue the strategy to enter each representation into the chart and then compare the displayed number. Since some representations require entering more than ten tokens for the same place value, at some point they start to utilize de-bundling to get the required quantity of tokens for each place value.

In the following scene Hanna tries to enter the representation “2 ones and 60 tenths” into the digital place value chart utilizing the de-bundling strategy for the first time.

3 Hanna: (*taps two tokens into the ones‘ column and one token into the tenths’ column of the digital place value chart on the iPad, she then taps four tokens into the tens‘ column and drags one of the tokens into the tenths’ column (100 tokens emerge within the tenths’ column), see figure 3*) Oh, I didn’t want that many. (*drags the three remaining tokens out of the tens’ column*)

Figure 3: De-bundling as dragging from tens to tenths

4 Bella: You can also drag it back (*points on the tenths' column of the digital place value chart on the iPad and then on the tens' column*) and then away.

5 Hanna: (*drags one token from the tenths' to the tens' column (100 tokens of the tenth merge together to one point within the tens'column), then taps one token into the ones' column and drags it to the tenths' column (10 tokens emerge within the tenths' column) and repeats this strategy another five times, finally she deletes the one token in the tens' and in the tenths' column*)

In this example we notice that connecting in terms of exploring the relation between tens, ones and tenths takes place to establish operational dragging on the concrete level of reference (line 3 & 5). The direct feedback of the place value chart enables this by initiating a reflection on de-bundling from tens to tenths (line 3) and then correcting the connection by dragging tokens from ones to tenths (line 5).

Operational dragging can also comprise dragging as a gesture executed either on the potential level of reference above the place value chart or on the free level of reference in the gesture space. In both cases it simulates the real dragging on purpose within the chart.

When the students are asked to write down the standard notation of the decimal number represented by "8 ones and 10 tenths", Bella suggests to enter this representation into the place value chart by the use of de-bundling.

6 Bella: Don't we wanna do it like this perhaps? (*points at the tens' column of the digital place value chart on the iPad*) so here a token (*moves the hand to the right above the ones' column, see figure 4*) and then remove two (*stretches the pointing finger two times*) I think that would be faster.

Figure 4: De-bundling as a dragging-gesture from tens to ones

In this example the dragging gesture to the right (see figure 4) can be characterized as a mismatch, because the dragging is only present in the gesture and completely missing in speech. Nevertheless, it may simulate the action of dragging on the potential level of reference and is afterwards performed by the students directly on the iPad (level of the concrete).

At the end of the second teaching experiment, the students are asked to describe the rules of the digital place value chart and to reflect on a meta-level about the decimal place value system. In this situation, the students see the consistent base-ten-property of the decimal place value system (cf. Ross, 1989) explaining bundling more generally than only referring to the relation between specific place values:

7 Bella: That te so ten than become (*puts her left hand fingers upright onto the table*) on the (*raises all finger except the thumb*) si on the le (*puts her left hand fingers upright again*) so o when we drag them more to the left (*moves her hand leftwards on the table, see figure 5*) always one (*raises and lowers her hand on the table*)

Figure 5: Bundling as a dragging-gesture from the right to the left

The halting verbal utterance explaining "bundling" is accompanied here by a dragging-gesture from right to left on the table without pointing directly or indirectly to any concrete representation, so that it can be dedicated to the free level of reference (line 7 & figure 5). In this phase of structure seeing, the base-ten property is generalized for bundling activities referring indirectly to the digital place value chart by the verbal description of "dragging" as well as by the dragging-gesture from the right to the left. This dragging-gesture can be characterized as *structural dragging* referring generally to bundling which is part of the place value system's structure.

As a conclusion we can state that the movement of dragging is preserved on all three referential levels of gestures and can be observed within the three epistemic actions. In particular, operational dragging is performed on all three levels. Therefore it could be characterized as a connecting mode which on the one hand refers back to practical dragging on the concrete level and on the other hand it could prepare structural dragging on the free level of reference.

DISCUSSION & OUTLOOK

In this study we have reconstructed three different modes of dragging that seemed to support students' constituting of the decimal's structure by working on particular tasks with a digital place value chart on the iPad. These modes have been detected in the epistemic process of a pair of students. This was done by matching the analysis of the epistemic actions, when dragging-movements were performed, with the analysis of the referential levels on which dragging took place. As a result we were able to distinguish and characterize practical, operational, and structural dragging by the ways they contributed to the students' constituting of decimal numbers as a structure. In the project DeciPlace, additional data will be used to explore the conditions under which these dragging modes emerge, how they exactly are connected, how they shape epistemic processes as a whole and in what way they lead to building a stable concept of decimal fractions, even for low achieving students.

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FEATURES OF THE DEEP APPROACH TO MATHEMATICS LEARNING: EVIDENCE FROM EXCEPTIONAL STUDENTS

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It is widely acknowledged that there are individual differences in the way students approach the learning process, and that these are reflected in the learning outcomes. Little research has been done from the learning approaches perspective regarding mathematics learning. We report an exploratory study investigating the features of the deep approach to mathematics learning. We present the case study of two exceptionally competent students who participated in an in-depth interview. Indicators of the deep learning approach along the categories Goals, Study/Learning strategies, Self-regulation aspects, and Motivation are presented. These findings can be employed in the design of instruments to be used in quantitative research.

THEORETICAL BACKGROUND

It is widely acknowledged that there are individual differences in the way students approach the learning process. A main distinction is the one of the superficial versus the deep approach to learning (Entwistle & McCune, 2004). The surface approach is associated with the intention to reproduce the content when necessary. On the other hand, the deep approach to learning is associated with the intention to understand, and is typically related to stronger conceptual understanding of the intended material as well as with higher performance (Chin & Brown, 2000; Chiu, 2011; Smith & Wood, 2000; Stathopoulou & Vosniadou, 2007).

An important question in this research area is the description of the features of each learning approach and their indicators (Cano & Berbén, 2009; Entwistle & Cune, 2004). This is particularly the case for the deep approach to learning, for which different researchers opt for different features and/or indicators. Moreover, empirical studies validating features and indicators have been conducted mainly with adult participants (typically university students). As Entwistle and Cune (2004) argue, however, the defining features of each learning approach cannot be generalized across different disciplines and age groups.

Research from the learning approaches perspective is scarce with respect to mathematics learning and focused mainly on tertiary education (e.g., Cano & Berbén, 2009; Smith & Wood, 2000; but see also Chiu, 2011, for a study with late primary school students).

In a previous study (Bempeni & Vamvakoussi, 2015), we attempted to capture the features of the deep and the surface approach to mathematics learning for secondary school students. Following Stathopoulou and Vosniadou's (2007) work on learning approaches to science learning with adolescent participants, we started with three

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 75–82. Szeged, Hungary: PME.

categories, namely Goals, Study strategies and Awareness of understanding. Our results showed that students with surface approach value school performance as a goal adopt memorization and rehearsal as study strategies and have low awareness of their understanding and of the effectiveness of their study strategies. On the contrary, students that follow the deep approach combine theory studying and extensive practice, invest time in mathematics studying on a long-term basis, and are highly aware of their understanding and of the effectiveness of their study strategies. However, our data indicated that there are other aspects of our participants' approach to mathematics learning that were not captured by our initial categories, mainly regarding motivational and self-regulatory aspects of mathematics learning and studying (see also Cano & Berbén, 2009; Entwistle, McCune, & Tait, 2013).

In the present study we attempted to detect and describe in greater detail the features of the deep approach in mathematics learning by studying exceptionally competent students in mathematics. We adapted appropriately our previous instrument and we enriched it with the categories Motivation and Self-regulation.

METHODOLOGY

Participants

The participants of the study were two students, one sixth grader (hereafter S_1) and one ninth grader (hereafter S_2) with exceptional competence in mathematics according to their mathematics teachers. We note that we didn't rely merely on this information; we also tested their conceptual knowledge in a specific content area, namely rational numbers.

Specifically, we used 25 tasks compatible with Rittle-Johnson and Schneider's (2014) categorization of tasks targeting mathematical conceptual knowledge: a) evaluate unfamiliar procedures, b) evaluate examples on concept, c) evaluate quality of answers given by others, d) translate quantities between representational systems, e) compare quantities, f) invent principle-based shortcut procedures, g) generate or select definitions of concepts, h) explain why procedures work.

These students dealt with these rather challenging tasks very successfully, indicating that they had deep conceptual knowledge in this content area, and also that they were highly competent in mathematical reasoning, in problem solving, and in explaining and justifying their reasoning.

Research instruments

We developed 29 items in the form of scenarios that the students had to react to (e.g., "*If you had to advise a younger student how to study mathematics what would you consider important to tell?", "You observe a friend of yours studying mathematics without solving exercises. You see him dedicate a lot of time studying the theory, making diagrams, going back to previous units, taking notes. Do you study mathematics in the same way? Is there any advice that you would like to offer?",* *"A younger student asks for your help with the comparison of fractions. What would you do to help him?*").

Procedure

The students participated in two in-depth semi-structured individual interviews. During the first interview, they were asked to solve the rational number tasks, thinking aloud and explaining their answers. During the second interview students were asked about their learning approach to mathematics. The second interview took place about three days later. Each interview lasted about one and half hours. All interviews were recorded and transcribed.

Data analysis

The starting points for our analysis were the following categories: a) Goals, b) Study strategies, c) Awareness, d) Self-Regulation, and e) Motivation. The indicators for each of the categories were: a) Understanding-Personal making of meaning, b) Combining theory and practice, Validation, Long-term time investment, Integration of ideas, c) High, d) Monitoring, Regulation, Control of cognition and emotions, e) Intellectual challenge, respectively (Bempeni & Vamvakoussi, 2015; Entwistle et al., 2013).

Features of the deep approach in the learning of mathematics	
Categories	Indicators
Goals	Understanding - Personal making of meaning
	Academic success
Study/Learning strategies	Active involvement
	Validation
	Combining theory and practice
	Long-term time investment – Solving unfamiliar problems
	Integration of ideas
Self-regulation aspects	Monitoring/regulation of understanding
	Awareness of the understanding and the effectiveness of one's strategies
	Regulation of emotions during the exam
	Regulation of study behaviour
	Flexibility in the use of strategies
Motivation	Intellectual challenge

Table 1: Features of deep approach in the learning of mathematics

We selected sentences as unit analysis, but in some cases we used paragraphs so as to obtain a sense of the whole. We looked for utterances that included keywords pertaining to the indicators of each category (e.g., understand, concept, meaning for the indicator personal construction of meaning). We placed the sentences in the coding categories according to the initial indicators and developed new indicators when needed. After coding, data that could not be coded were identified and analyzed later to determine if they represented a new category. New indicators emerged for the categories Goals, Study strategies, Self-regulation and Motivation.

In addition, we merged the categories Awareness and Self-regulation in one more general category namely Self-regulation aspects because in our data utterances related to awareness and self-regulation typically were intertwined. The categories are presented in Table 1.

RESULTS

Goals

Both students stressed that they care about marks, and also for their teachers' and schoolmates' opinion. However, they also stressed the importance of understanding in mathematics and especially of personal making of meaning.

> [Mathematics] is not rote learning. The point is to try on our own and understand. If I could not cope with mathematics and the teacher graded me higher than I deserved, I would try more. Mathematics is a useful subject and I have to understand it. [...] Fractions do not only relate to comparison rules. First of all, you must understand what fraction is. If you do have everything in your mind and know what a fraction represents, then it is easier to solve what you are asked and to consider fractions much more familiar. (S_1)

> Mathematics is not like other subjects that you have to memorize things-you must put your mind to the work, think sensibly. I prefer discovering new things on my own, because in that case I will never forget them. [...] There are other ways to compare fractions except rules. You do have to understand the fraction as quantity, to represent it with a figure. Estimating, using common sense... (S_2)

Study strategies

Active involvement. Both students indicated that they are actively engaged in learning in the mathematics classroom: they recognize what they do not understand, do not hesitate to express their questions and assess the information they receive.

> [A good student] is not reluctant to express and support his/her opinion. […] When I don't agree with my teacher I always step up. For example, I could not understand why we cannot use decimals as fraction terms. Since the fraction represents a division, why is it not allowed to use decimals as numerators or denominators? Decimals can also be divided, can't they? (S_1)

> Once I had doubts about what my teacher said. But I dared to express my objection and we had a scientific debate. I gave it up only when I realized that I was wrong. However, sometimes I happen to be right. (S_2)

Validation. As we can conclude from the above mentioned transcripts, the two students are not willing to accept something if it is not sufficiently proved. At different points of the interview, they mentioned that they use «common sense» to check their results or to monitor their steps while solving (see also transcripts in the section «Self-regulation aspects»). We note that both students monitored the solution process during the first phase of the study (e.g., they used counterfactual proof).

Combining theory and practice. Both students referred to the importance of combining deep understanding of theory and solving exercises.

> Mathematics is theory too, if you don't understand the theory well you cannot solve problems. There is always some theory behide the problem. How can you solve a problem with proportions if you have no idea of what proportion is? You must also practice with many exercises. But learning the rules by heart does not help. Then, in problems, how can the rules be useful to me? Will I simply write down the rule? (S_1)

> Both theory and exercises are important. If you do not study theory, you cannot solve but only the simplest problems. (S_2)

Long-term time investement – Solving unfamiliar problems. Both children appeared to value the long-term time investment on mathematics studying.

> It is necessary for students to do extensive practice in mathematics, because when gaps are created, it is quite difficult to understand the more advanced material. That is why I try to solve many exercises by myself except the ones I have for homework. (S_1)

> Studying should not be restricted to what is required in the course. […] I do a lot of practice during the private tutoring lessons I attend. (S_2)

For these students, practice is not limited to the study of solved examples or to solving similar problems.

> Solving many similar exercises is not enough. Then if you are asked to solve a slightly different problem, you cannot do it. This is because you can deal only with similar problems, with different numbers. I think that if somebody has not understood the material, then they cannot think further and solve unfamiliar problems. (S_1)

> I do not like solving similar exercises all the time. Repetition may be useful for other subjects but not for mathematics. For example, I do not believe that memorizing the solutions of exercises in mathematics is useful even if one can solve them when asked. (S_2)

Integration of ideas. Both students referred to importance of making connection among different units of mathematics and also relating mathematics to other subjects (Physics, Chemistry), and to everyday life, too.

> Yes, I think that the previous and the following units are connected in some way in mathematics. For example, we had been taught proportions and then percents, for which good knowledge of proportions was necessary. And if you want to understand proportions well, you need to understand fractions as well. (S_1)

> We were taught the distributive property with numbers when we were at sixth grade, and then we were taught the same property with variables, and the same holds for all other properties. (S_2)

It is worth mentioning that both students valued the connection of different representations in mathematics, and also to everyday life as an appropriate instructional method.

> Teachers need to make mathematics real for students, to show mathematics in real life. For example when we say ¼ kg cheese what do we mean? How much is it? (S_1)

> In the first years of school-life students have not understood fraction as quantity. I could help a younger student to understand it with figures and representations. (S_2)

Self-regulation aspects. Both students appeared to monitor, control and regulate themselves in the level of cognition, emotions and behavior in the learning of mathematics.

> When I face a difficulty, I try to see the problem from many different aspects and construct a table with what is given and asked in my mind. You can be aware if the process goes well while solving, if you monitor what you're doing and do not solve it mechanically. You can also verify by putting numbers in case you want to make sure that you are correct. I validate in my mind without making operations. You should also pay attention to the result, the result should be reasonable. (S_1)

> I am sure that I have understood the problem, when I am able to put that in my own words, when I have the problem in my mind and it is not necessary to read it all the time. When I have difficulty in understanding the problem I break it into small parts and then I try slowly to understand what I do not do well. (S_2)

As a result, both students appeared to have a high awareness of understanding and to be able to differentiate the difficulties in understanding from the school requirements.

> At first, I found fractions a little bit difficult. Not the operations and the exercises, these were very easy. (S_1)

> I did not understand the unit "probability" that we were recently taught. I found it disjointed but I tried to understand using paper and pencil. […] Understanding the concepts that you are now taught in a greater grade is something very usual. For example we are accustomed to «cross-multiply». But you should look into it deeper, so as to understand the algorithm. (S_2)

What is also notable is their reference to the way they face an unfamiliar problem in the exam context.

> You have to try until the last moment. If you make negative thoughts from the very start, then you will not solve the problem even if you possess the sufficient knowledge to do it. If you have time, you can try until the end. There is no reason to give up. (S_1)

> At first you say, "Oh my God", then you are starting to swear, and finally you say "I will do my best. I will not die, after all, it's just a test! (S_2)

Both students appeared to recognize that the combination of insistence on trying and flexibility is necessary.

> Once I had difficulty with a problem in a test I left it last. When I came back to it, I tried to look it from another perspective. Generally, when I realize that my method is not efficient, I try to apply some other knowledge, even if I am not sure that this is the correct way to solve the problem. (S_1)

> I simply made different thoughts. And when my thoughts took me nowhere, I rejected them. I thought different things regarding the solution and I got rid of the ones that did not help me. (S_2)

We note that S_1 stated that she is always concentrated when studying so as to need less time. S_2 «revealed» that he started courses with a personal tutor because he wanted somebody to motivate him to do more practice. Both students showed self-confidence regarding their current learning strategies in mathematics. S_2 mentioned that he did not pay attention to the theory in the past and he added: «I realized it later, but I do not believe it was late». Moreover, he referred to his strategy focusing more on exercises and stated: «I understand in my own way. If I realize that this way is inappropriate, I will change it».

Motivation. Both students appeared to be motivated by unfamiliar and challenging problems.

> I prefer problems that are difficult, when you need to think of something by yourself. I don't like the ones that are solved in a particular way, mechanically. I find all these exercises with tables that we do the method of cross-multiplying all the time very boring. (S_1)

> I find uninteresting what keeps me from going further. Everything that has operations and you must do constantly the same. That's why Geometry is a more interesting part to deal with. (S_2)

CONCLUSIONS-DISCUSSION

This exploratory study investigated the learning approach to mathematics of exceptionally competent students, with the intention to trace features of the deep approach to mathematics learning. The results provide indicators along the categories Goals, Study/Learning strategies, Self-regulation aspects and Motivation (Bempeni & Vamvakoussi, 2015; Entwistle et al., 2013).

More specifically, the two students value the personal making of meaning, without neglecting academic success. They invest time in the study of mathematics, and consider the solving of unfamiliar problems an important part of practice. Despite the fact that they recognize the value of the theory, they do not dedicate much time to study it. This inconsistency may be explained by the quality of participation in the school classroom which is a central learning strategy for these students. They also actively look for connections among different representations, content units, different subjects, and everyday life. Validation of mathematical knowledge is highly significant for them: they actively seek for validation in the school context and when they solve problems. Furthermore, both students monitor, regulate, and control their emotions and their behavior in the context of mathematics learning and studying. As a result, they are highly aware of their understanding and their learning strategies, and they are flexible. Finally, they are motivated by intellectual challenge.

The findings of the present study offer a more detailed insight into the features of the deep approach to mathematics learning and can form the basis for the design of a research instrument to be used in quantitative studies. It should be noted, however, that these findings also point to the fact that the construct "learning approach" is rather broad and overlaps with other constructs stemming for other research perspectives (e.g., "intentional learning", "self-regulated learning" – for a similar observation see Cano & Berbén, 2009). More detailed analysis of such constructs is necessary to highlight possible similarities and differences.

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READING AND LEARNING FROM MATHEMATICS TEXTBOOKS: AN ANALYTIC FRAMEWORK

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I present a framework within which to analyse ways in which university students (in this case, teachers learning mathematics content) read and study mathematics from a standard PreCalculus textbook. In terms of both the course design and this research, the teachers are conceptualised as learners. The framework examines the relationship between the written discourse (the mathematics in the textbook) and the enacted discourse (the student's utterances as she reads the text). Theoretically this relationship is understood in terms of commognition. Sub-categories of interaction are derived inductively from observations of five students each participating in a talk-aloud session in which they read and study mathematics text. Sub-categories of interaction are illustrated through typical exemplars from two talk-aloud sessions.

Mathematics textbooks have the potential to be an accessible and powerful resource in and out of mathematics classrooms. Yet very little is known about their use by learners (Osterholm & Bergqvist, 2013; Rezat, 2013). At the tertiary level, within which my research is grounded, Shepherd, Selden and Selden (2012, p. 226) argue that "it appears to be common knowledge that many, perhaps most, beginning university students do not read large parts of their mathematics textbooks in a way that is very useful in their learning. Whether this is because they cannot read in such a way or choose not to do so does not seem to have been established". In this paper, I hope to contribute to this important research by presenting an analytic framework within which to analyse university students' reading of mathematics in textbooks.

THE ANALYTIC FRAMEWORK

I present an analytic framework based on two pillars: the discourse of the mathematics textbook (the written discourse) and the ways in which students interact with this discourse (the enacted discourse). The theorised sorts of interaction between these two forms of discourse derives largely from Sfard's theory of commognition (2008) whereby individual development is theorised as individualization of collective activity and thinking is theorised as individualized communication. In mathematics, collective activity takes the form of mathematical discourse in which a range of permissible actions and re-actions are apposite. Learning takes place through the individual's thoughtful participation in mathematical discourse, such as the discourse in a mathematics textbook. Mathematics learning is indicated by a change in the individual's mathematics discourse.

The analytic framework is refined through my observation of five individual students as they studied a particular section of a mathematics textbook (the written discourse) using a talk-aloud protocol (the enacted discourse).

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 83–90. Szeged, Hungary: PME.

Written Discourse: The different components of a traditional mathematical textbook are described using Sfard's (2008) description of mathematics as a discourse**.** This discourse is characterised by its narratives, routines, visual mediators and words. Narrative is text that is used to define or describe or justify the existence of mathematical objects, for example, theorems, definitions and proofs; routine is text that refers to activities with these objects, e.g. worked examples, exercises; visual mediators are the symbols and notation of mathematics as well as alternate representations of mathematical objects, e.g. graphs, diagrams. These three categories of discourse all employ words in a particular way, and thus, for the purpose of this paper, I do not distinguish 'words' as a separate category. The instantiation of these characteristics, i.e., examples, exercises, proofs, theorems, graphs and so on, are regarded as mediators. They give mediated access to the mathematics objects that are created, acted upon and engaged with in mathematics discourse. See Berger (2015).

Enacted discourse: The enacted discourse describes ways in which the students interact with the written discourse. Given that my research concern is with the ways in which students use or do not use the textbook to promote learning, the categories for analysing *how* the student uses the textbook whilst studying from it, are related to the ways in which students engage with or ignore the different components in the textbook. Broadly speaking the categories are: using the textbook; injecting knowledge from other sources; making connections between different mediators in the textbook. These broad categories are then inductively (Hatch) refined using empirical data from the five students' interviews.

THE CONTEXT

The study took place within the context of a PreCalculus module. This module was part of a one-year postgraduate degree (Honours level) programme in Mathematics Education at a South African University. The module consisted of one 3-hour session per week for eleven weeks. I was the lecturer in charge of the module.

South African high school mathematics teachers often have weak mathematics specific content knowledge. This is partly a result of teachers often having a degree or diploma in education and an absence of traditional university-level maths courses (a legacy of apartheid). Accordingly the focus of this course was on deepening and broadening mathematics content knowledge. For this reason, the in-service teachers were conceptualised as mathematics learners in the design of the course, and in this research paper. Another important aspect of the course was an emphasis on self-learning. This emphasis derived from the idea that learning to use texts and other resources to learn or re-learn mathematical topics is necessary for a teacher who needs to keep on expanding and deepening her mathematical knowledge. In order to foreground the practice of self-learning, at the beginning of the course, all students (in this case, the teachers who are conceptualised as learners) were given a hand-out in which they were told exactly which part of the prescribed textbook (Sullivan, 2012) to study for each weekly session. As they were repeatedly told, studying involved carefully working through definitions, theorems and proofs and worked examples of the text, referring to other resources or other section of the textbook if they so required, and doing and handing-in a set of prescribed exercises (taken from the back of the chapter). Thus 'new' mathematical knowledge or revisited mathematical knowledge was accessed by the students through the prescribed textbook **prior** to the lecture. In order to encourage this self-studying before class, students were given a multiple-choice test on the prescribed study material at the beginning of each session. Throughout the course, the lecturer (myself) discussed ways of reading mathematics text, for example, ways in which to deconstruct definitions. But these ways were based on my own experience rather than on research related to how students do read textbooks.

RESEARCH FOCUS

What analytic framework can be used to understand how students use a prescribed mathematics textbook for learning new mathematics and deepening knowledge of previously-encountered mathematics in a self-study mathematics course?

COLLECTION OF DATA

Six students from the PreCalculus module (two low-performing students, two medium-performing and two high-performing students) were asked if they were willing to be video-taped while studying a prescribed section of a chapter (the sub-chapter) from the standard prescribed textbook. Five agreed. In terms of the expectations of what this studying entailed, and as discussed with the students in weekly sessions, studying meant carefully reading through the sub-chapter including definitions, theorems and proofs, referring to other resources or other sections of the textbook if they so required, working through worked examples and doing a set of prescribed exercises contained in the sub-chapter. For the video-taped session, students were asked to study the sub-chapter 'Properties of Logarithms' (Sullivan, 2012, pp. 296–304) as they would in preparation for class and to talk out loud as they did so. They were also given a set of exercises at the back of this sub-chapter as was the case for their weekly sessions. This sub-chapter was chosen because it was not part of the course and it involved both new content and content which they had previously encountered (in school and at undergraduate level) but from a more advanced perspective. For example, these students should have known the Sum of Logs property but they were unlikely to have ever proved it. The 'new' knowledge included some new properties of logarithms and their proofs. Each video-taped session was at most 1.5 hours long. Before, during and after the studying of the sub-chapter, students were asked to point our which sections were new or familiar to them.

Coding the data

The main analytic categories were: Textbook Opportunities (using the textbook); Injections (injecting knowledge from other sources); Connections (making connections between different mediators in the textbook).

Textbook Opportunities: Episodes in which the student explicitly used the textbook to look at specific routines, narratives or visual mediators but with no indication as to whether this looking was productive or not, were coded OPT. Episodes in which the student did not use the textbook to clarify, illuminate or enrich their understanding of the mathematics discourse, despite such content being available in the textbook, were coded as 'missed opportunities' (OPTM). If the student used the textbook productively, for example, to make explicit connections, to generalise, to exemplify, etc. this was coded as OPTP. If the student attempted to use the textbook to enrich understanding of an aspect of math discourse but looked at discourse (e.g. theorems, worked examples, etc.) which were **not relevant** to the issue at hand, this was coded as OPTU. Finally, if the student looked at appropriate theorems, worked examples and so on, but was unable to see the connections to their particular conundrum or difficulty, this was coded as OPTX.

Injections: If the student injected discourse from a source other than the textbook into the reading of the text, this was coded as INR or IND depending on whether the injection was 'robust' or 'distractive'. These terms (injection, robust and distraction) were introduced by Leshota (2015) in her analysis of school mathematics teachers' use of textbooks as a resource for teaching. Robust refers to productive insertions of discourse; distractive refers to unproductive or confusing insertions. If it was clear that the injection was that of prior knowledge, the code INPR or INPD (robust or distractive respectively) was used; if the injected discourse derived explicitly from the interviewer or a computer or resources other than the textbook, the code was INIR or INID (depending on whether it was robust or distractive).

Connections: Since this framework is around how students engage with the discourse of the textbook, it was important to look at what connections (MC) were made both within the textbook and between textbook discourse and other discourse. And so there are codes for making connections between different parts of the text (MCT); making connections between interviewer's interjections and text (MCI); making connections between prior knowledge and text (MCP); making connections between visual mediator (eg a diagram) and text (MCV).

ILLUSTRATION OF CATEGORIES AND CODES

For purposes of illustration I look at exemplars of some of the more commonly used sub-categories in the analytic framework. These exemplars are taken from two contrasting students. Abby, is a very high-performing student; she has an undergraduate degree in science with two years of Mathematics (taught by Science Faculty) and a postgraduate diploma in education; she has been teaching at high-school level for just over two years. Tom is a relatively low-performing student. He had graduated with a B.Ed degree (which involved some mathematics courses taught by the education faculty) in the previous year and had some maths tutoring experience in the School of Education.

Exemplar 1: OPTP and MCT

Abby's reading of the text is comprehensive: she reads the entire text section, including all the different theorems, proofs and worked examples, before attempting any exercises. Throughout this reading she makes a large number of connections to prior knowledge and to other parts of the sub-chapter. In the vignette below, we see Abby (A) engaging with the given proof (Figure 1) of a property (Property 6) which she has not encountered previously. She uses the textbook productively, reading carefully through the proof of the property and making explicit the properties that this proof uses, and which she has just read (Figure 2). For example, she writes $a^{\log_a M} = M$ (which is Property 1) on the text next to the first line of the proof. That is, she makes overt connections between the proof of Property 6 and other given properties. Accordingly the vignette (Table 1) is coded OPTP and MCT.

> **Proof of property (6):** From property (1) with $a = e$, we have
> $$e^{\ln M} = M$$
> Now let $M = a^x$ and apply property (5).
> $$e^{\ln a^x} = e^{x \ln a} = a^x$$

Fig 1: Textbook Discourse – Proof of Property 6 (Sullivan, p. 298.)

> For a, M positive real numbers, $a \neq 1$, r any real number.
>
> Property 1: $a^{\log_a M} = M$; Property 2: $\log_a a^r = r$;
>
> Property 5: $\log_a M^r = r \log_a M$; Property 6: $a^x = e^{x \ln a}$

Fig 2: Some properties of logs given in textbook.

What is said and done	Reader Activity	Comment on Reader Activity	Code
A: I'm focusing more on proof six now. So they mention property one again over here. So again, just to clarify everything in my head, I would go back and check property one. (*Pages back to find Property 1.*) So I like to have that with me. So any time they mention that they're using property one and it's something I haven't seen before, I generally write property one next to it (*writes)*. So that I can see alright. So they said that a is going to equal e, and obviously when you're using e, you're going to have to change it to ln. So I'm just identifying why they've used e, then ln n, and then M.	A writes $a^{\log_a M} = M$ next to $e^{\ln M} = M$ in textbook**.**	Makes connections between new narrative (proof of Property 6) and previous narratives (Property 1 and Property 5)	OPTP MCT

And then just spotting the differences. And then they're saying, *M* now must be what we want up here, *a* to the *x*. (*Reading*) Okay, so just taking a look at how they go from *e* ln *a x*, and then they take the *x* to the front and then that becomes property five, which just becomes *a* to the *x*.			

Table 1: Enacted Discourse - Exemplifying OPTP and MCT

Exemplar 2: INPR

Abby is doing Exercise 22 (Figure 3). She solves the exercise seamlessly, referring to previous experience as she does this (Table 2). Hence the vignette is coded as INPR.

In Problems 13 – 28, use properties of logarithms to find the exact value of each expression. Do not use a calculator.

22. $\log_a 16 - \log_a 2$

Fig 3: Textbook Discourse – Exercise 22 (Sullivan, p. 303)

What is said and done	Reader Activity	Comment on Reader Activity	Code
A: Okay, then we're on 22, it says, do not use a calculator, which also would give it away. So this is…okay, now this is again is something that I've done with my kids, so it would be quite easy. Minus, then you just go and you say, it's log…	Solves exercise problem correctly	Reader refers specifically to previous knowledge of this sort of exercise.	INPR

Table 2: Enacted Discourse - Exemplifying INPR

Exemplar 3: Exemplifying OPT, OPTU and INPD

On starting the session, Tom skims over the narratives and routines (including theorems, proofs and worked examples) and plunges into the exercises. From time to time, while doing the exercises, he refers back to worked examples (WE) and occasionally to statements of properties of logarithms (but he never reads the proofs). His engagement with the narratives and routines is characterised by a lack of attention to detail. In the vignette below (Table 3) Tom gets stuck on the simplification of terms involving logs and powers, while doing a routine, Exercise 52 (Figure 4).

Although Tom is able to correctly simplify, using prior knowledge, the given expression (coded INPR) to $\log x^3 + \log(x+1)^{\frac{1}{2}} - \log(x-2)^2$, he is unable to simplify further (to write powers as logs). Accordingly he turns to a different routine, WE 2 (Figure 5) in textbook (OPT). This routine explicitly uses Properties 1 and 2 which precede the example. In this episode, Tom is referring to discourse (a worked example) which is not directly **relevant** to the issue at hand. Hence this is coded as OPTU. Also Tom is using (incorrect) prior knowledge (INPD). See Table 3.

Q52: Write each expression as a sum and or a difference of logarithms. Express powers as factors: $\log \frac{x^3\sqrt{x+1}}{(x-2)^2} \quad x>2$

Fig. 4. Textbook Discourse (Sullivan, p.303): Exercise 52

Example 2: Using Properties (1) and (2) (***a***) $2^{\log_2 \pi} = \pi$ (b) $\log_{0.2} 0.2^{-\sqrt{2}} = -\sqrt{2}$ (c) $\ln e^{kt} = kt$

Fig, 5: Textbook Discourse (Sullivan, p. 298): Worked Example 2

What is said and done	Reader Activity	Comment on Reader Activity	Code
T: Yes, ma'am. It's example two, on page 298.	Looks at WE examples silently	Looking at similar routines.	OPT
T: And then they say, log of this can be…but these are all of base ten (referring to exercise that he doing)…so this is log of *x* cube plus log of *x* plus one to the power of two, minus log of x minus two. Okay, well, I don't know if that…they're all of the base ten, so I cannot apply the rule that says log…that, says that I should come and multiply say log three log *x* or half because they're all of base ten. They don't have different bases here (*referring to WE 2*) so I don't think it's allowed to apply that function. I would leave it at this stage.	Indicates that he cannot use WE 2 because in WE 2, the bases were the same as the expression whereas in Exercise 52, the different expressions all have base 10.	Does not realize that Property 5 (Narrative) – see Figure 2- is applicable to the simplification required for Exercise 52 (Routine).	OPTU INPD

Table 3: Enacted Discourse – Exemplifying OPT, OPTU, INPD

CONCLUSION

In this paper I have described a broad framework within which it is possible and feasible to analyse students' reading of mathematics textbooks in a self-study context. I have illustrated the framework through a sample of purposefully chosen exemplars from specially designed talk-aloud sessions. The value of such a framework, grounded in both theory and the empirical, lies in its potential use as a tool to better understand how university students read mathematics texts for self-study; such an understanding could lead to explicit tutoring in useful ways of reading mathematics texts. Being able to read and learn from mathematics texts is surely a very important skill in the learning of mathematics.

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RESPONSES TO "THE SCARY QUESTION": HOW TEACHING CHALLENGES IMPACT THE USE OF KNOWLEDGE AND ITS DEVELOPMENT

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This paper reports on teachers' experiences of being out of their comfort zone in their mathematics teaching. We describe examples of experiences that the teachers considered "scary", their reported responses to those situations, and the longer-term effects of such experiences. Implications for the acquisition of knowledge for teaching mathematics are discussed, and questions raised about the possible impacts of confidence and experience on the interaction between discomforting experiences and teacher learning.

Considerable effort has been made to characterise the knowledge required to teach mathematics. Since Shulman (1987, p. 8) described pedagogical content knowledge (PCK) as "that special amalgam of content and pedagogy that is uniquely the province of teachers" PCK has been a focus of mathematics education researchers who have conceptualised Shulman's notion of PCK in a variety of ways in relation to mathematics teaching (e.g., Chick, Baker, Pham, & Cheng, 2006; Hill, Ball, & Schilling, 2008; Rowland, Huckstep, & Thwaites, 2005).

There are many complexities associated with PCK and its study. The role of content knowledge and generic pedagogical knowledge as contributors to PCK is acknowledged but not always well articulated. For example, there have been extensive efforts to identify mathematical knowledge for teaching (MKT) (e.g., Hill et al., 2008). The framework of Chick et al. (2006) attempted to identify key aspects of PCK on a continuum reflecting the mutual entailment of pedagogical knowledge and content knowledge, and showing how these broader constructs combine and interact to impact on PCK. To add to the complexity, some characterisations of PCK view it as a body of knowledge, aspects of which are either held or not (e.g., you either know how to respond to a particular student misconception or not). This suggests that PCK is measurable, and although it acknowledges that PCK can change, in a given moment—such as at the time of being measured—it is static. Other characterisations consider PCK dynamically, notably in the ideas of transformation, connection, and contingency in the knowledge quartet of Rowland et al. (2005). This emphasises "knowledge-in-action", and recognises knowledge-based decision-making that takes place in response to unexpected classroom occurrences.

The question of how best to examine teacher knowledge is thus difficult to address. Large-scale written "tests", such as those used by Hill et al. (2008), cannot capture the rationale that teachers might have for their choices. Surveys with follow-up interviews (e.g., Chick et al., 2006) may provide insights into the knowledge that informs

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 91–98. Szeged, Hungary: PME.

teachers' choices and responses, but still cannot capture the nuances of decision-making in the complex milieu of the classroom. Classroom studies (e.g., Rowland et al., 2005) are labour intensive, and may capture the actions of a teacher but not always the underlying activated knowledge that motivated those actions.

All the characterisations of teacher knowledge implicitly recognise that it can change. There has been only limited research into how teacher knowledge develops: what circumstances provoke change and growth, and the extent to which teachers seek to change. Chick and Stacey (2013) report on two occasions where Chick, as a teacher educator, appeared to develop PCK. The first involved her trusting that she would have sufficient mathematical expertise to respond appropriately during an unexpected and unprecedented class diversion; the second provoked her to rethink her expectations and actions during in-class problem-solving situations.

More attention has been paid to how the beliefs of teacher might be changed. Although much has been written about the difficulty of influencing teachers' beliefs, Liljedahl (2010) described five mechanisms by which teachers beliefs can change. Liljedahl and others (e.g., Rolka, Roesken, & Liljedahl, 2007) have drawn upon theories of conceptual change as well as Green's (1971) metaphorical description of belief systems. According to the latter, beliefs may or may not be evidentially held and, to the extent that they are, evidence that contradicts them may result in change. Many beliefs, however, that are non-evidentially held are among the most strongly, or centrally, held being tied up with teachers' identities. These are among the most resistant to change. At other times, the juxtaposition of contradictory beliefs, previously held in discrete clusters (Green, 1971) that have prevented their contradictory nature from being recognised, can be a catalyst for change. Change is at least dependent upon recognition of a need to change (e.g., Liljedahl's (2010) examples), the availability of a plausible alternative, and willingness to make the required effort (Arvold & Albright, 1995).

The distinction between knowledge and beliefs is, of course, not clear-cut. Beswick, Callingham, and Watson (2012) provided evidence of a rich construct of knowledge needed for mathematics teaching that encompassed Shulman's knowledge types, as well as confidence and relevant aspects of beliefs. In light of this, and the examples from Chick and Stacey (2013) regarding how confidence influenced a response to an unforseen teaching incident, it seems reasonable to infer that much of the literature on belief change might also be applicable to the development of teachers' knowledge. The purpose of this paper is to examine how teachers address these "scary" areas, and how this affects their knowledge for teaching mathematics.

THE STUDY

The study was part of the first phase of a larger study of the knowledge needed by teachers of mathematics and English. Phase 1 involved focus groups with experienced primary and secondary teachers in two Australian states (Tasmania and Victoria), and New Zealand (NZ). The focus group discussions examined the kinds of knowledge that experienced teachers use in their teaching, and how they acquire that knowledge. The particular focus of this paper is on how the uncomfortable or challenging experiences of teaching reveal and affect the development of knowledge for teaching mathematics.

Participants

The teachers who participated in the study were invited because researchers or colleagues knew them to have the experience and expertise needed to be able to discuss the knowledge that teachers use in teaching mathematics. There were three focus groups used for this study, one each for primary and secondary teachers (each with three or four teachers), and a single Victorian focus group (the sole secondary teacher joined the three primary teachers for most of that discussion). In all, there were 11 teachers of mathematics, of which six were primary teachers and five were secondary.

Instrument and procedure

The focus groups were intended to uncover the knowledge of expert teachers, via questions addressing a range of issues related to mathematics PCK. Prompts included examples of student misconceptions, sets of tasks to stimulate discussion of task sequencing, and examples of activities. The discussions were conducted by different members of the research team (the authors, Rosemary Callingham, and Tim Burgess) and were audio-recorded for later transcription. They lasted between 75-105 minutes. The discussions were semi-structured with questions provided as a guide but they largely took the form of collegial conversations between the teachers and researchers.

Initial versions of the interview protocols were used with Tasmanian focus groups, and then revised. During their discussions the Tasmanian teachers mentioned the effects of being out of one's comfort zone when teaching. This led to the development of "the scary question" which is the focus of this study. It was included on the protocols used with the Victorian and NZ teachers who comprise the participants for this study. The question was: "If you were to be in a position where you had to teach something that was out of your 'area', how would you go about it? You may like to name what your 'scary space' might be". In some cases the teachers were prompted to consider how they would respond to having to teach mathematics that they had not taught before or at a grade level much higher or lower than those with which they had experience.

RESULTS

The transcripts were examined for responses related to the scary question and also for discussion in other parts of the focus group discussion that dealt with the experience of being challenged in one's mathematics teaching. Most of the relevant discussion was in response to the scary question. Responses fell into one of three categories:

1. Identifying a scary space in mathematics teaching,
2. Describing teacher responses to being in such a space, and
3. Considering the longer-term consequences and/or implications of such experiences.

In most cases the challenges identified, responses to them, and longer-term consequences were provided in the context of stories of personal experience. There were only occasional mentions of the challenges faced by other teachers, typically inexperienced or "out of area", and how they might respond. Table 1 summarises the responses from the primary teacher focus groups in relation to the first two of these categories. Specific stories that are reflected in Table 1 included the following:

> I had to teach Year 12 … just as a relieving period, and the kids are like, "How are we going to solve this?" And I said, "Where are the answers?" I found the answers, […] I worked backwards, and I said, "for me, I don't know, off the top of my head how to solve this, but let's start at the end, and work out the process". (NZ primary teacher)

Identifying	Describing responses to the situation
Multiplying and dividing fractions	Go back over the mathematics myself
Quadratic equations	Think about my understanding and how it makes sense to me
Being a relief teacher for a Year 12 mathematics class	Consult an expert in person (advisor, academic, colleague, son)
"Where I don't know the maths"	Consult an expert via email
Other teachers who are frightened by the maths, don't like the maths	Being unafraid to ask
	Use YouTube and other internet resources e.g., online tutorials
	Ask the students what else they could do besides asking the teacher to explain
	Work backwards from the answers with the class
	Consult a mathematics teacher education text
	Do problems myself before the lesson
	Use own difficulties and errors to stimulate class discussion
	Learn with the students

Table 1. Responses of primary teachers to the scary question

Table 2 summarises the responses of the secondary focus groups in relation to the first two categories. This group shared fewer specific stories than the primary teachers; one example was the following:

> I was the only teacher, but I had some cross-over with statistics, the Year 13 full course, so we went to any PD we could, we worked together, and, I also worked with the year below teachers, to get the next the level three, so, we worked below and aside, and we looked at all the paperwork, everything we could online, you know […]. But it was very collaborative. (NZ secondary teacher)

The NZ secondary teachers also reported positive impacts of teaching challenges, saying that successfully facing such situations encouraged them to be more innovative and inclined to take risks in their teaching. They reported positive responses from students to changed and "risky" pedagogy even if there was initial resistance. For example, one teacher described how he had got his students to mark their own exams. Although there were protestations about this being his job not theirs, the students performed the task honestly and thoughtfully and, according to the teacher, learned a great deal in the process. There was agreement in the group that teaching outside of one's comfort zone improves pedagogy. One expressed it as follows:

> If you've been teaching for a long time … you do end up being on auto-pilot a little bit. You know where the pit-falls are and that they'll misunderstand this one, and all of those sort of things, but when you've got something new you can't afford to be in that situation, so, in a funny way, it probably actually improves classroom teaching, it does for me. (NZ secondary teacher)

Identifying	Describing responses to the situation
Networks Level three statistics – new concepts, broad statements, lack of resources Using problems you don't know the answer to Student resistance to innovation Using problems without scaffolding Moves to International Baccalaureate curriculum	Avoid teaching the topic Allow students to work alone Work through with a colleague, "nut it out" together Take professional development opportunities Read online material Work with teachers of the year below Work with teachers in another school Ask students to explain thinking If students are getting things right don't worry about how (if you can't follow their reasoning) Persevere with innovation despite student resistance Get as much information as possible – from people rather than the internet Find out how to introduce the topic Other teachers might use textbooks

Table 2. Responses of secondary teachers to the scary question

Another teacher in that group explained that not knowing the answer to a problem "forces you to have those conversations [asking them about their thinking] with kids rather than knowing the answer and standing up the front." They agreed that changing school, curriculum, or colleagues, or undertaking post-graduate study can all cause teaching "to evolve". This evolution included a greater inclination to take risks:

> I've taken more risks in the classroom in activities … I completely changed the approach that that I would normally have taken, and I tried something. (NZ secondary teacher)

The Victorian secondary teacher explained how the introduction of the International Baccalaureate curriculum was forcing some teachers to change. She said:

> … it gives the teachers no choice then, they have to actually use the tasks … It's sort of a very sledge-hammer approach though, it's not ideal, … you know, there's a balance between trying to bring people on board with it, and look, there's been a lot of positive out of it, a lot of people saying, "Oh, I never thought". (Victorian secondary teacher)

The only negative consequence of the uncomfortable situations they described was that less confident teachers who do not like mathematics or those teaching out of area tend to fall back on traditional and safe teaching approaches including reliance on textbooks and algorithms, and telling students.

DISCUSSION

The discussion is framed around the two foci of the study. The first deals with what a focus group discussion about uncomfortable teaching experiences can reveal about teachers' knowledge; the second considers how the teachers' responses suggest ways in which such experiences can influence the development of knowledge for teaching.

Revealing knowledge through discussing uncomfortable teaching situations

An immediate effect of facing a challenging teaching situation is recognition that one's existing knowledge is not sufficient for the task. For the experienced teachers in this study this was accompanied by an assessment of the relative importance of the knowledge they needed to acquire and of how necessary or urgent its acquisition was. The teachers were able to recall and carry out a range of possible strategies for acquiring the needed knowledge. These included seeking out colleagues, internet and textbook resources, and professional learning opportunities. This suggests that knowledge of how to expand one's own knowledge may be an important part of knowledge for teaching mathematics. Importantly, these teachers also evidenced an inclination to learn, underpinned by implicit confidence that they could.

The scary scenarios that the teachers cited were teaching problems that demanded solving, and so provide contexts in which the teachers acted as problem solvers. Indeed the steps of Polya's (1957) heuristic can be identified in the processes they articulated: understanding the nature and extent of the problem (their lack of knowledge); making a plan to solve the problem; carrying out the plan; and reflecting on the effectiveness of the teaching that followed. Chick and Stacey (2013) characterised mathematics teaching as a problem solving activity in which teachers bring to bear their mathematical and pedagogical knowledge in order to solve a mathematics teaching problem. In the challenges these teachers described, however, the problem was more often concerned with acquiring needed mathematical knowledge rather than applying it. This highlights a tension: the initial "scariness" is often a lack of knowledge of mathematics, but after the requisite mathematical knowledge is acquired, what are the implications for the development of the relevant necessary PCK? Given the claims of the specialised nature of PCK for mathematics, it is not likely to be enough to bring general pedagogical practices to bear, so there must be a quest for further knowledge.

The scary question also appeared to be effective in revealing the more innovative pedagogies in the teachers' repertoires. It prompted them to recount stories of the use of pedagogies that were inherently risky: where they were not sure of being in control to the extent that were possible with more familiar approaches, nor sure of the ways in which students would respond, or certain of the direction in which the lesson might go.

Development of knowledge for teaching

Facing uncomfortable experiences in teaching mathematics appeared to have had positive effects on the knowledge development of these teachers. We know that significant belief change requires both willingness and effort on the part of teachers (Arvold & Albright, 1995) and it seems reasonable to extend this to changes in knowledge more generally. The innovative practices that the teachers described were largely precipitated by being in situations in which familiar strategies were judged not viable and hence trying something new was the only option. Without the impetus of a challenging situation it is possible that innovative pedagogies known to teachers may not be used. The scary situations appear to have prompted thought and experimentation with pedagogical practice, influenced by the content to be addressed, leading to the possible growth or at least broader understanding of PCK. These situations had also prompted teachers to reflect on their current knowledge and actively seek new knowledge. Sources of this knowledge included the internet and texts, but there was a clear preference for learning with colleague teachers, advisors or academics. For some teachers, more formal activities such as participation in professional learning opportunities or postgraduate study were ways of growing their knowledge.

In contrast to their own experience, the teachers in this study were less positive about the impacts of challenging teaching situations for less experienced or less expert teachers such as those fearful of mathematics or teaching out of field. Rather than improving the teaching of these teachers, the experienced teachers believed that being uncomfortable about mathematics teaching resulted in textbook-reliant teacher-centred practice on the part of those less expert. Of course, teachers teaching out of field or who have negative attitudes to mathematics may find that every mathematics lesson that they teach is a scary experience. If impression of the teachers in this study is correct, it could be that always being in a scary teaching space rather than experiencing these situations only occasionally is a disabling rather than enabling experience.

The underlying confidence of the experienced teachers in their capacity to acquire the knowledge that they needed and to deal with pedagogical challenges is also likely to make an important difference. This confidence may lie in either or both of content and pedagogy; the leap into the unknown described in one of the examples in Chick and Stacey (2013) was attempted because of the teacher's confidence that both her mathematical knowledge and general pedagogical knowledge would be sufficient to deal with what arose. Just as in teaching mathematics the level of challenge presented by a task needs to be appropriate to the learners (Wertsch, 2011), it could be that for experienced teachers the need to teach unfamiliar mathematics is seen as an achievable challenge whereas less experienced or expert colleagues might experience it as overwhelming.

CONCLUSION

Posing the scary question prompted teachers to reflect on teaching situations in which they felt inadequate. This methodological approach appears to be a useful technique for uncovering the more innovative pedagogical approaches that the teachers employ in teaching mathematics and hence for gaining a fuller picture of the PCK that they have at their disposal. It also suggests that a relevant aspect of teacher knowledge not included in current models could be knowledge of ways to gain further knowledge of mathematics, pedagogy, or PCK. The outcomes also suggest that finding themselves in challenging teaching situations can make a positive contribution to the knowledge development of teachers, with the possible caveat that the degree of challenge corresponds appropriately with the teachers' existing expertise and confidence.

Acknowledgement. Research funded by ARC Discovery Grant No. DP130103144. The contributions of Rosemary Callingham and Tim Burgess are also acknowledged.

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TEACHING FUNCTIONS IN A SECONDARY SCHOOL

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Abstract: This paper presents a multi-case study of teachers describing and justifying the ways in which they teach functions at secondary school level. Typical forms of the reported teaching practice are identified; they show that despite the strong influence of the school vision on teaching mathematics, there is still space for individual variation. This variation is based on the differences in the teachers' epistemological models of functions linked to their visions of learning mathematics.

INTRODUCTION

As early as 1991, Dreyfus (1991, 20) has ascribed functions as one of the "most difficult concepts to master and teach in all of school mathematics" (p. 120). The function concept encompasses many layers of complexity and sub-concepts (cf. p. 120) such as variables (Malle, 1993); correspondence and covariation (Malle, 2000); covariation related to rate of change (Johnson, 2015); and graphical representations (Stölting, 2007; Vogel, 2007). This complexity causes a variety of mistakes and misconceptions about functions that students show at the end of secondary school level (Nitsch, 2014). Representational fluency (cf. Suh & Moyer, 2007) seems to be underdeveloped. Hence, students are burdened with many learning difficulties at the initial state of high school (the *Oberstufe).* In this state, the topic of functions is readdressed to prepare students for calculus, there we have observed an additional problem in Germany: The kinds of mistakes, and the unfavourable ways in which students deal with functions seem unpredictable broad. In order to design suitable lessons for this student body, teachers have to know how their students have been taught functions previously. But this is difficult to achieve, since the students come from many feeder schools with a variety of different school cultures in which they may have been taught functions in many ways, with different purposes and emphases. However, beyond the curricular steps of types of functions provided by textbooks, little is known about such variable teaching practices. This problem is addressed in an ongoing qualitative empirical study that investigates teaching practices on functions at the secondary school level. This paper reports on an initial approach which answers the following research question: What kinds of vision on teaching functions in the secondary school can be reconstructed at the institutional school level and at the teachers' level. Preliminary results will be presented involving seven teachers, five of whom come from a reform school.

THEORETICAL FRAMEWORK: THE CONCEPT OF PRAXEOLOGY

Ways of teaching functions are determined by matching two perspectives, a perspective on institutional practices based on school programs and their visions and on the individual teacher's view of the function concept and how it is, should or can be

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 99–106. Szeged, Hungary: PME.

taught best. This matching is conceptualized in the concept of “praxeological equipment” which stems from the *Anthropological Theory of the Didactic* (ATD) and will now briefly be described.

All human activity consists of practice and discourse; Chevallard talks about *praxis* and *logos* (Chevallard, 2006, 23). These two directions of acting are shaped by two blocks, the practical and the theoretical block. The practical block consists of tasks to be conducted and techniques for how these tasks as ways of doing are or can be carried out. The theoretical block consists of technology and theory, describing the kind of discourse on the techniques and its theoretical basis. Technology is the way in which the techniques enacted by the tasks are described, justified, and validated (see Bosch & Gascón, 2014, 69). The theory consists of the basic assumptions underlying the technology. ATD uses the term praxeology which is shaped by the quadruple (task, technique, technology, theory).

According to the topic of functions, the *bath task* is an interesting task which is currently used in German schools. It considers a bathtub filled with water and a graphical representation of a story: It asks students to invent a possible story that fits the representation. For that, a number of techniques may be considered, for example, the slope of the graph, sections where it is positive, zero or negative, using the variables in the coordinate system, addressing points on the graph, and so on. The kind of reasoning in the description of the story would tell us about the technology used; for example, the students could explain the story either by contextualized or de-contextualized language, going back and forth from the story to the graph. The underlying theory of mathematics could be that mathematics provides tools for interpreting real-world stories by the use of mathematical concepts.

Didactic praxeology involves “setting-up a praxeology” (Chevallard cited by Bosch and Gascón, 2014, 69). In our case it describes the activity through which the mathematical praxeology on functions is implemented in class as an institutionalized way of doing mathematics. If we consider the ‘bath-task’ to be implemented in class as a didactical tasks of a teacher, this could be justified by pointing to the relevance of diagrammatic reasoning on covariation and the use of modelling an everyday situation, the first aspect to provide access to covariation as an important feature of functions and the latter to link this concept of functions to the students’ empirical world. Such a justification expresses a didactic technology and highlights the assumption that mathematics can best be learned if it is related to the students’ experience.

When a teacher talks about the way he/she teaches functions then we gain information about a mixture of institutional didactic praxeologies and the teachers’ individual views on teaching and learning functions, on teaching mathematics and on pedagogy in general. With reference to Chevallard, Bosch and Gascón (2014, 69), we call this mixture “praxeological equipment”. Students do not only get used to or adopt institutional praxeologies as a habit but build their own praxeological equipment that is also influenced by those of the teachers. Therefore, the secondary teachers’ praxeological equipment is an important factor which may direct students' interpretation of task requirements in the initial phase in high school.

METHODOLOGY, METHOD AND TECHNIQUES

The empirical study presented here is part of a design-based research project called "The function concept in the transition to high school", conducted by the interdisciplinary research group FaBiT (Fachbezogene Bildungsprozesse in Transformation, see Doff et al., 2014)* at the University of Bremen, Germany. One aim of this study is to reconstruct teachers' praxeological equipment for teaching functions at secondary school level. This equipment is the result of the school's praxeology being modulated by the individual teachers' views and biographies. Therefore, the methodical approach will be twofold: (1) the didactic praxeology on teaching functions will be reconstructed by analyzing the school program, talking to the educational chair, and by ethnographic documentations through participant observation; and (2) traces of teachers' praxeological equipment are reconstructed, based on interviews. The interviews are meant to be conversations producing narratives that disclose official and implicit personal views and experience. They last about 60 minutes and are partially structured following these subsequent questions: How do you teach functions, what is relevant, on what do you pay attention, and which resources or teaching aids do you use (e.g. the textbook)? Provide a task that is typical for your way of teaching functions and explain your choice. How do you teach formulas and how are they linked to functions? Since there is not one single function concept being taught in school, it is important to know the teachers' epistemological models of the function concept. Therefore, we have asked additionally: What are functions for you?

This paper focuses on an empirical multi-case study of one reform school. The data consists of five audio interviews with teachers (we refer to them by abbreviations beginning with T for teacher: TUB, TIB, TFB, TOB, TEB) from the reform school including the educational chair, the school program (available on the school's homepage) and ethnographic documentations on ways in which teachers can access to tasks and task descriptions. To be able to make distinctions, these five interviews are complemented by and contrasted with additional interviews with two teachers (TO1, TO2) from two other schools. All interviews are audio-recorded; pictures are taken of the tasks provided by the teachers.

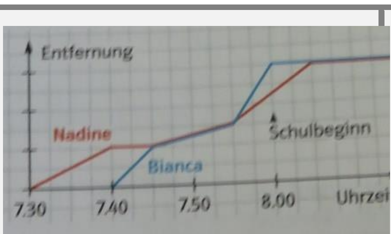
According to the praxeological categories, the school program (cf. homepage) is analyzed and complemented by participant observations. The audio-recordings of all seven interviews are analyzed in three steps: (1) identifying relevant categories in the audio recordings and transcribing the interview scenes around them, (2) distinguishing the pronouns *we, us,* and *our* for institutionalized practices and *I, me,* and *my* for individual practices; and (3) reconstructing the teachers' typical praxeological equipment and their epistemological models of functions. Finally, all analyses are matched.

PRAXEOLOGICAL ANALAYSES ILLUSTRATED BY SOME DATA

Tasks and techniques

The school was established as a learning community where teachers work together, prepare and develop lessons in class-grade teams not only for themselves but also for their colleagues, who may adapt and improve the tasks in the following year. This is expressed by teachers using the words “we, us, our” but also through stressing the “open door conception”: If doors to the classrooms are open, colleagues are invited to participate (cf. TOB). Tasks are gathered in boxes or uploaded into a Dropbox with descriptions (cf. TOB). Parallel tests are written regularly (cf. TUB) and taken as a starting point to analyze weak and strong points to be discussed and improved. In this way, the teachers have discovered that improving reasoning should be a major issue (see TIB). These techniques establish a common vision of teaching.

The program of mathematics is arranged around topics from the textbook (cf. TUB). Learning arrangements are commonly developed (see TIB, TEB) and strengthen the common vision. When teachers experience teaching aids as effective (TIB, TEB), they voluntarily share the school vision on which the aids are based (“the tasks work very well” (TFB)). Through this program and working in class-grade teams, new teachers are quickly introduced into the school practice of teaching mathematics (cf. TFB). In the school vision, the function concept is a core concept (TIB, TUB, TOB). In the interviews, teachers talk about teaching functions in different teaching topics such as teaching scales, fractions, check lists as well as tables and graphs from the beginning in grade five (TIB, TOB, TUB, TFB). Nearly all teachers have pointed to similar tasks as being typical for teaching functions in this school: An early up-take of distance-time-graphs in grade 5 (Fig.1) to be taken up and deepened in grade 7 (Fig 2).



x-axis: time, y-axis: distance School starts.

Tell stories about Nadine’s and Bainca’s routes to school and show this in the diagram. (mathe live 5, p. 70, own translation)

Fig.1: Typical task example for grade 5

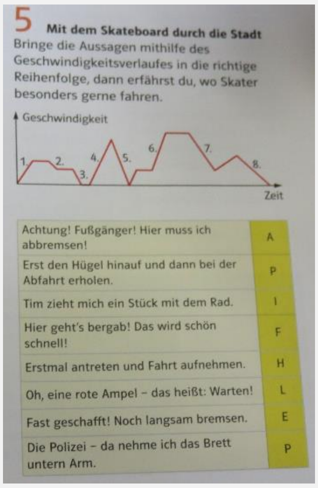


Fig.2: Task for grade 7

Skating down town

Arrange the statements referring to the velocity in the right way, then you will know where skaters like to drive.

A: Watch out! Pedestrians. Here I have to brake.

P: First up-hill and then relaxed down-hill.

I: Tim pulls me a bit with his bike.

F: Now down-hill. This will be wonderful speedy.

H: First step up and then gather speed.

L: Uh, a red traffic light- that means waiting.

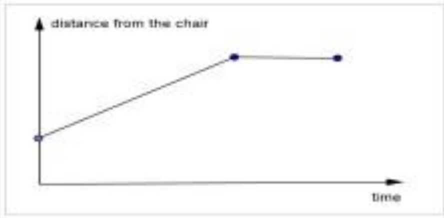
E: nearly managed! Only a slow brake

P: The police – there I take the board under the arm. (mathe live 7, p.73, own translation)

Fig. 3: Task example for grade 7/8

1. In the boxes with the same color is the same number of matches.
2. On both sides of the equation sign are the same number of matches.

How many matches are in the boxes? (mathe live 7, p. 153)



One walks, the other draws the graph.

Fig. 4: Walking graphs

Technology and theory

The vision of teaching and learning mathematics is available on the homepage: The main ideas are that all students participate in ambitious, competence-oriented learning and all students are supported in the ways they need: instruction is regarded as contemporary and modern, arranged around topics which come from the students' empirical world, varied and vivid, but also addressing competitions and out-of-school learning. Concepts are built up in a spiral curriculum, thus appearing again and again (ed. chair). All teachers emphasize that understanding is more important than the ability to use symbols, e.g. TUB has expressed this point by saying: it is always important to know "what stands behind, what the slope means" (1.23, TUB). This school's didactic vision serves as theoretical background for practice, which is reflected in the educational chair's narrative but also referred to by the teachers to justify the choice of tasks and techniques. For example, teachers point to "use functions flexibly … in various application contexts" (2:00, TUB), and refer to the task in Fig. 4 as a tangibly "experience [of] functions" (6.50, TFB). However, the teachers also modulate the didactic praxeology at the school level in their praxeological equipment.

Praxeological equipment of the teachers

Technology and theory of the teachers' didactic praxeologies are based on individual values and preferences, reflecting two aspects: the *second subject they teach* (arts, geography, language, science), and their *epistemological models of functions*. In the five interviews from the reform school, we found, apart from nuances, three types of traces of such didactic praxeological equipment:

Type a: *Preparing equations: Teaching functions means teaching equations since functions only come into existence with variables, terms and equations*. Everything that is learned before is only preparation. Even graphical representations only prepare functions. This also means that sub-concepts such as the concept of variables must be built up before, otherwise "the function concept cannot be understood" (4.0 TEB). This is done best when *students directly experience concepts*; e.g. the perimeter can be walked in footsteps but this is not possible for the area, hence, going along the perimeter initiates the experience of the difference between area and perimeter (13.TEB).

Type b: *Providing application contexts*: A *function is a correspondence between two magnitudes (cf. Thompson 2008) that is expressed by contextual stories, tables, graphs and other representations*. Therefore, *teaching functions means flexibly changing representations and contexts*. This can be carried out with climate diagrams ("as a GUB teacher [society and politics], [I find] climate diagram are exciting" (3.50, TUB)), or motion stories. "a highlight with which we have made positive experience, are distance-time-diagrams ...implemented early in grade 5 ... that is, motion stories" or talking about "routes to school" (2.55, TUB). (Fig. 1, 2, 4).

Type c: *Nurturing progression*: *Functions are tools for analysing and predicting factual connections*. This can best be achieved when students answer their own questions (3.00, TIB) and are able to interpret functions (4.30, TIB). *Teaching functions means providing meaningful contexts, starting early by inquiry learning, and progressing by connecting content across the grades* (1.42, TIB). Thus, concrete tasks such as measuring the child's room or inventing motion stories (Fig. 1) provide access. Friction occurs when functions are expressed by terms and equations. But tasks like "Knack die Box" (Fig. 3, 8.10, TIB) may provide "tangible experience" (4.30, TIB) out of which students' own questioning may still occur again. The next two types are expressed by the two teachers from other schools. Their didactic praxeologies are shaped around different visions of instruction:

Type d: *Generalizing models: Teaching functions means "experimenting with functions as tools for modelling"* to prepare high school mathematics right from the beginning, as co-varying *changes of magnitudes*. A typical task is the bath task. Examples for instance tables from science are used as a didactic technique to introduce functions. But examples are not enough to access the shape of the function as a general structure. Predictions provide the need for a more general view leading to a structure such as a formula. In order to avoid friction and to keep understanding lively, contexts and mathematical expressions should always be related. (cf. TO2)
This is the only teacher who has high school requirements in mind. We believe that this is the case because he also teaches at the high school level and the others do not.

Type e: *Teaching parts: Teaching functions should be done by teaching the functions' main parts*, e.g. "the m [slope] and the n [y-intersect]" (TO1) in the use of linear functions, and how they can be found and drawn. *Functions are built by their main parts; for linear functions this is the slope and the y-intersect*. Using graphs means finding points on the graphs and drawing graphs means drawing points; didactic techniques are worksheets and examples. Structuring instruction is done by designing clearly arranged worksheets. The categories of evaluating tasks and worksheets are "simple" or "difficult" (TO1). Simple examples and simple language are used to make tasks accessible, hence, simpler. The main aim is to avoid difficulties. (cf. TO1)
The narrative of this teacher is reduced to describing techniques. The underlying assumption that guides technology seems to be that tasks 'looking simple and manageable' (TO1) will provide access to mathematics, avoid overexertion, and help students overcome problems.

Role of the teachers' epistemological models

In contrast to the last two teachers, the teachers in the reform school did not talk about specific examples, neither as a step towards generalizing nor to simplify the tasks. They were more concerned with how the function concept could be built up as a whole idea right from grade five by engaging the students in many types of diagramming. However, their epistemological models of the function concept differ and deeply influence the way they talk about tasks, or justify their choice and techniques. For example, if functions come into existence only as equations (as in type a), then the term *function* does not appear when using graphs or tables in grades 5-7, or when graphical representations cannot be described by equations. Teacher TO2 teaches also at the high school level (type e), therefore, he seems to be concerned with developing competences for this level. In this respect, the awareness of generalizing seems to be an important point. According to his view on functions as a modelling tool, generalizing is meant to be described by functions as a dynamic structure allowing predictions and extrapolations, e.g. in science (his second subject). Analysing and predicting can also lead to another teaching view. TIB's theory is concerned with the students' factual connections (type c) and their own questions as a fruitful recourse. TIB admits that this can lead to friction when algebra begins. His technique is to find tasks that overcome this friction, such as the Knack-die-Box task, which may initiate students' own questioning through inquiry. If a function is predominantly seen as a correspondence between specific magnitudes in contextual situations being expressed by representations (type b), then it is clear that this correspondence can, as expressed by TUB, best be "addressed by the flexible use of representations".

In contrast to the four holistic views on functions, type d presents a fragmented view of functions. When simplifying tasks is an additional technique to make tasks accessible, students will have difficulties building up functions as entities made of many parts.

CONCLUDING REMARKS

To date, we have only reconstructed traces of praxeological equipment of seven teachers from three schools providing a small excerpt of how praxeological equipment might constrain students' learning during the transition to high school. When students adopt a *fragmented view of functions* they will have difficulties as soon as a flexible use of function is required in high school. When students adopt the view that *functions only come into being when they can be described by an equation,* they will also encounter problems as soon as functions need more than one equation to be expressed (e.g. the absolute value function) or cannot be described by equations at all. The other views of teaching functions differ only slightly; together they emphasize that functions should be taught as early as possible (25.30 TIB), by *meaningful contexts, nurturing progression* and *generalizing models* to prepare for high school level. A more formal definition of types of functions might then be a natural extension of what is already known leading to a pattern for predictions and extrapolation. (cf. 10.50, TIB).

* Funded by the Excellence Initiative of the German Government (www.uni-bremen.de/cu-fabit).

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BORDER CROSSING: BRINGING TOGETHER PRE-SERVICE TEACHERS' TECHNOLOGICAL, PEDAGOGICAL, AND CONTENT KNOWLEDGE THROUGH THE USE OF DIGITAL TEXTBOOKS IN MATHEMATICS

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Although technology has long been identified as an integral part of mathematics education, there has been, more often than not, an overemphasis on the use of technology as a goal rather than a means to learning mathematics. Thus, researchers are still grappling with how to integrate technology that is, at once, subject-based as well as compatible with learning theories and pedagogical strategies. In this paper, the authors use the theoretical model TPACK to demonstrate a mutually interconnected framework. Data were collected from two courses in a teacher education program in mathematics where the instructor (the first author) used a grade-six digital textbook to teach the pre-service teachers. Three examples are discussed to illustrate the use of TPACK and its implications for further research.

INTRODUCTION

School mathematic has traditionally been associated with textbooks and curriculum material (Remillard, 2005). But, with the increasingly growing awareness of the importance of technology in mathematics education (Drijvers et al., 2010), and with the realization that the use of technology helps leaners achieve a better understanding of mathematical concepts and their relationship (Hohenwarter, Hohenwarter, & Lavicza, 2008), technology too has been gaining recognition in the teaching of school mathematics. Having said that, researchers have been witnessing several problems in relation to subject-based, pedagogically sound integration of technology into teaching and learning. Most recently, Ornellas and Sancho-Gil (2015) have warned against treating the mere availability of technology as a panacea. Obviously, it is not. To date, with the prevalence of technology in mathematics education, teachers have a plethora of digital applications, digital labs, videos, and recommendations for teaching the subject content. Nonetheless, Brush and Saye (2009) found that teachers feel inadequately prepared for subject-specific use of technology. It is thus understandable why adapting subject specific, technology-based teaching techniques continues to be a concern for teacher educators (Lagrange, Artigue, Laborde & Trouche, 2003). To address this concern, researchers called for a theoretical framework to be used in incorporating technology into teaching (Brush & Saye, 2009; Kramarski & Michalsky, 2010). The purpose of this paper is to answer this call by employing the TPACK

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 107–114. Szeged, Hungary: PME.

model—an acronym for technological, pedagogical, and content knowledge—that we elaborate on in the next section.

THEORETICAL BACKGROUND

Technological knowledge, content knowledge, and pedagogical knowledge

We draw on Lundeberg, Bergland, Klyczek, and Hoffman's (2003) TPACK model, which offers an integrated approach that simultaneously encompasses three distinct education-specific domains: knowledge of technology, knowledge of the subject content, and knowledge of pedagogy (Niess et al., 2009). Table 1 provides definitions of the different domains of TPACK.

TPACK domain	Definition	Example
TK	Knowledge about how to use equipment or software	Knowledge about how to use Facebook, WhatsApp
PK	Knowledge about theories and teaching methods	Knowledge about Socio-cultural theory
CK	Knowledge about the subject content	Knowledge of mathematics
TCK	Knowledge about how to use technology to teach specific subject content	Knowledge about how to use GeoGera
TPACK	Knowledge about what technology to use and when	Knowledge about what technology to use to best teach area of a circle

Table 1: Definitions of TPACK domains. Adapted from Chai, Koh, and Tsai (2013)

In this paper, we focus on the interplay of the three dimensions in TPACK in the context of a teacher education program in mathematics. We also note that our attention was drawn to the intersection of knowledge of technology, content knowledge, and pedagogical knowledge rather than to the skill of using a particular application, software, or hardware. To wit, our aim is to explore the use of technology, content knowledge, and pedagogical content knowledge in a context of a teacher education program in mathematics.

RESEARCH QUESTIONS

The following research questions were formulated:

(1) How and to what extent do the TPACK domains intertwine when teaching and learning mathematics using digital books in a mathematics-teaching program?

(2) How do pre-service mathematics teachers put forth pedagogical considerations in a content-based environment of technology?

(3) What possibilities and limitations are attributed to the experience of using digital textbooks in mathematics education?

DATA SOURCES

In this study, the first author taught courses in elementary school mathematics to 26 pre-service teachers of mathematics using *Paths*—the Grade 6 digital textbook in geometry developed by the Centre for Educational Technology in Israel (Ovodenko et al., 2009). The digital textbook includes, among other things, an integrated forum platform, evaluation tools, and applications. The theoretical framework of TPACK was used throughout the duration of the courses to design multiple opportunities to solidify knowledge of technology, pedagogy, and content knowledge among the pre-service teachers of mathematics. To keep track of emergent needs, interesting scenarios, relevant examples, and theoretical and practical interconnections, the first author kept a journal of his experience teaching these courses. In addition to the course instructor's journal entries, the pre-service teachers' work, assignments, and activities were also recorded. TPACK-integrated manifestations were identified and analysed—three will be discussed in the next section.

DATA ANALYSIS

The following manifestations of TPACK were empirically derived through an examination of the pre-service teachers' work and the journal the first author kept. Each is framed as a learning opportunity that encompasses knowledge of technology, pedagogy, and mathematics. Whereas all manifestations include all three elements of TPACK, some include some elements in a higher degree than others. In the first example, we present a learning opportunity that exemplifies the interrelationship between knowledge of technology and pedagogical knowledge. The second example illustrates how knowledge of technology and content knowledge intertwine. The third example demonstrates the interplay among knowledge of technology, pedagogical knowledge, and content knowledge.

Learning opportunity 1: TPACK in action - Pen & paper or digital technology?

In grade six, one of the demonstrations used to teach the theorem of the area of a circle is an activity of cutting up a circle into regions and finding the relationship between the area of the regions (in approximation to the rectangle) and the area of the circle. In this context, pre-service teachers were asked to weigh the pedagogical possibilities and limitations of either literally cutting up a circle into regions or of using digital technology by merely pressing a button and seeing the circle unfold into a rectangle.

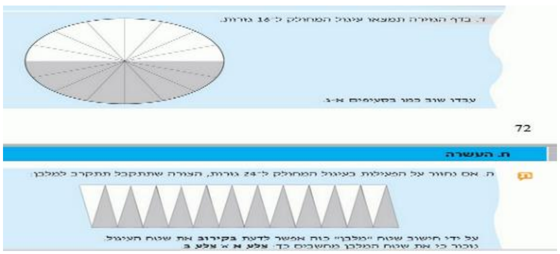


Figure 1: Carving up a circle into regions to find the relationship between the area of the regions and the area of the circle. From: Geometry for Grade 6, Paths, CET (Ovodenko et al., 2009)

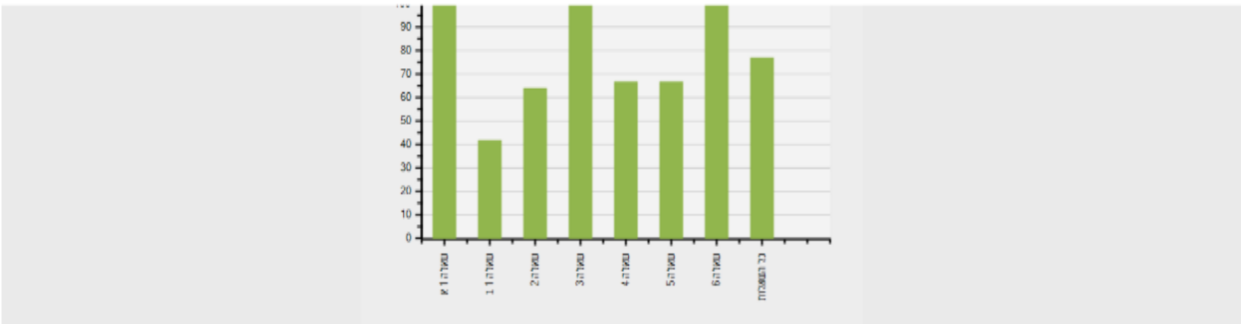
Here, the pre-service teachers were engaged in making a pedagogical decision. Which tool should they ask their students to use? What might students gain and what might they lose by working with one technology over the other? Pre-service teachers were invited to discuss these questions in a forum that was positioned right "on top" of the digital page and in juxtaposition to the assignment everybody was working on. The forum reflected pedagogical considerations the pre-service teachers weighed in as they were pointing to advantages and limitations of each activity. For example, one of the teachers explained:

> Because the student is the one who manually cuts up the circle and builds up the rectangle, it might be easier for him/her to make the connection between the radius of the circle, its circumference, and the length of the sides of the rectangle. [As for using the digital tool], there is the advantage of accurate use of the ruler that is integral part of the rectangle that's created. The app is suitable for children who are better at abstract thinking. As well, it is suitable for children who love to work on a computer. It is also suitable for children who have motor difficulties.

This excerpt demonstrates the interconnectedness of all three elements as pre-service teachers simultaneously use their knowledge of technology to express their pedagogical considerations and to implement subject-specific knowledge.

Learning opportunity 2: Real-time feedback (TCK)

Working with digital textbooks in mathematics allows teachers to provide real-time feedback. Each pre-service teacher sits at a computer and answers assigned questions from the digital textbook. In our context, as soon as the pre-service teachers submit the answers, an analysis of the responses is generated (see Figure 2) to capture how each of them did. This tool in the digital textbook allows the identification of content that requires further attention.



ציון מקסימאלי (באחוזים)	ציון מינימאלי (באחוזים)	סטיית תקן (באחוזים)	ציון ממוצע (באחוזים)	משקל השאלה	מספר משיבים	שאלה	
100	100	0	100	12	6	6i006d - גופים בחתכים: מקרה של חרוט	שאלה 1 א
100	50	29.48	41.67	12	6	6i008e - גופים בחתכים: מקרה של פירמידה מרובעת	שאלה 1 ב
100	33.33	23.65	63.89	12	6	6i006f גופים בחתכים: מקרה של פירמידה משולשת	שאלה 2
100	100	0	100	16	6	6i041 - גלידה לחברות	שאלה 3
100	100	23.57	66.67	16	6	6i042 - כיצד נראה החתך בפירמידה משולשת?	שאלה 4
100	100	23.57	66.67	16	6	6i043 - חתך של קובייה	שאלה 5
100	100	0	100	16	6	6i040 - "חותכים" את הציור	שאלה 6
100	33.33	24.87	78.98	-	6	כל השאלות	ממוצע

צפה בהתפלגות הציונים לפי כיתות ותלמידים

Figure 2: A snapshot of the success rate on an assignment generated by OFEK—a tool integrated in the digital Geometry textbook

The following is an example of an episode recorded in the instructor's journal. The pre-service teachers were asked to indicate True or False for the statement: "Two intersecting circles with different radii will always have two points of intersection." As the responses were streaming in, the course instructor (the first author) noticed that one of the pre-service teachers concluded that there are multiple intersection points between two intersecting circles. He next asked the student to draw what she meant (See circles on the left-hand side in Figure 3.). By this time, other pre-service teachers in the course also concurred that two intersecting circles may have multiple points of intersection. At this point, the course instructor explained that the use of a thick-lead pencil to draw the circles yielded this misperception. He then used GeoGebra to redress this misperception by using the zoom-in and zoom-out features embedded in GeoGebra to demonstrate to the pre-service teachers that intersecting circles with different radii, will always have two points of intersection (See circles on the right-hand side in Figure 3).

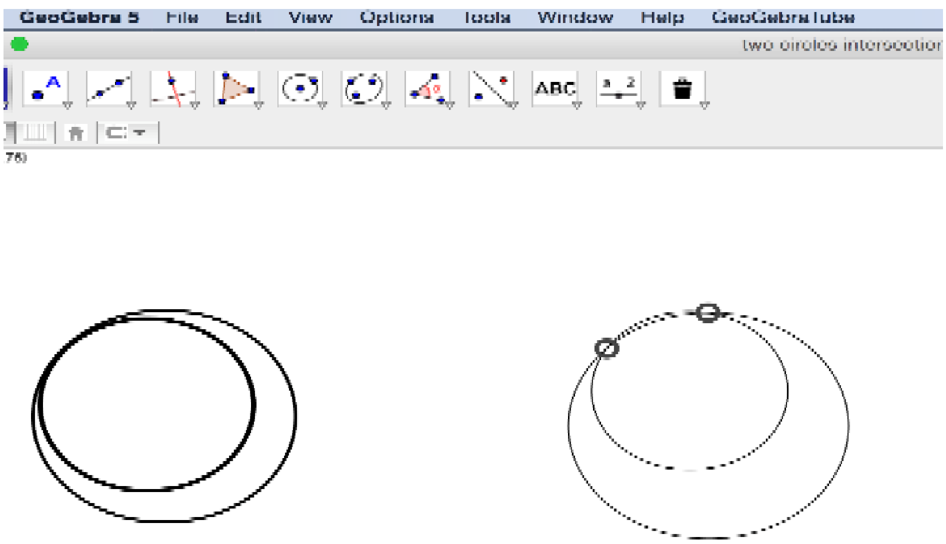


Figure 3: The response of the pre-service teacher and the course instructor were compacted for this paper into figure 3.

Learning opportunity 3: TPACK mutually interconnected

One of the features of the digital textbook allows teachers to upload additional assignments, tasks, or elaborations onto relevant pages thus simultaneously applying knowledge of technology, pedagogical knowledge, and subject content. In a topic about time units, a pre-service teacher incorporated a succinct summary of the time units: hour, minutes, and seconds, and asked the students to add three more examples for each time unit.

Figure 3: A pre-service teacher of mathematics uses the platform of the digital textbook to design and incorporate her own assignment within the book.

The rich variety of possibilities offered by the digital textbook allows the teacher to consider different pedagogical principles and a variety of digital tools to choose from.

DISCUSSION AND CONCLUSIONS

The three examples illustrate a variety of learning opportunities that were occasioned through the use of TPACK as a framework in the teacher education programs in mathematics. Working within this framework, there is added value in employing the forum tool (example 1) that can be plugged in within the textbook on a specific page, next to a specific section. Such a tool not only facilitates a visually available continuity in the exchange of content-specific pedagogical ideas, but also creates equal opportunities, for all, to participate regardless of considerations of wait-time, airtime, or turn-taking that typify non-digital platforms of discussion. In this study, the forum was task-specific. It was in fact located on top of the relevant page in the digital book and right next to the assignment thus creating a continuity of discourse that the pre-service teachers could refer to if they wished to better understand the pedagogical considerations of other pre-service teachers.

The second example highlights real-time feedback within the framework of TPACK. Real-time feedback that is occasioned within the framework of TPACK is, by default, content specific, prompt, and focused. It thus plays a critical role in student learning. The impact the feedback had on the pre-service mathematics teachers was significant as it yielded a learning opportunity which could have otherwise been gone unnoticed and hence overlooked.

The third example illustrates how digital textbooks allow for the integration of pedagogical considerations, content knowledge, and knowledge of technology to take place through the input of the mathematics teacher. While there may be limitations to using digital textbooks in teacher education programs of mathematics, we believe that the possibilities embedded in using the TPACK framework outweigh them. For example, using digital textbooks in a mathematics education program not only reduces costs by not having to provide hard copies of textbooks for all pre-service teachers to work with, but also allows the levelling of the playing field for all pre-service teachers so that they can collaboratively delve into the curriculum, improve their content knowledge, explore pedagogical practices, and hone their skills in digital technology. To wit, we demonstrated how harnessing TPACK as a framework can—when appropriate—provide instantaneity, accuracy, and perceptivity in the context of mathematics education. The analysis of the examples discussed in this paper demonstrates how the use of TPACK merits attention from the perspective of pre-service teachers of mathematics.

Bearing these comments in mind, the findings of this study may serve three purposes: provide evidence of how technology, pedagogy, and content knowledge intertwine and are in fact distinct but inseparable in the context of mathematics education; indicate TPACK's theoretical and practical relevance that pertains to the essential role of technology in the development of mathematical understanding; and suggest that TPACK is relevant not only in teacher education programs in mathematics but, by extension, also in professional development courses catered to in-service teachers. For the purposes of this study, we challenge the understanding of the TPACK framework as compartmentalized entities. We suggest treating these entities as distinct but inseparable dimensions that can be used as helpful tools to further explore how, through the principled consideration of the interplay between these dimensions, learning opportunities are occasioned.

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MEASURING LANGUAGE-RELATED OPPORTUNITIES TO LEARN IN PRIMARY MATHEMATICS CLASSROOMS

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Lower mathematical achievement of learners with a migration background was repeatedly explained by language proficiency. This study focusses on language-related opportunities to learn (OTL) during instruction, distinguishing active and receptive language-related OTL. An instrument to measure OTL was developed, evaluated and studied for similarities and differences between N=383 third graders from German and non-German speaking homes. Results show lower receptive, but higher active OTL of students from non-German speaking homes. Receptive OTL is positively related to mathematics skills in both groups, while active OTL only in learners from non-German speaking homes, pointing to the relevance of OTL for acquiring and fostering mathematics skills.

INTRODUCTION

International and national school achievement studies repeatedly revealed differences related to a migration background, indicating a disadvantage of students with a migration background in language skills, but also in mathematics achievement. In that research, a migration background is usually defined by the child's or parents' country of birth. For many of the about 35% of students with a migration background in Germany, the language spoken at home differs from the instructional language German that is spoken in school. There is evidence that language skills explain migration-related differences in mathematics achievement to a large extent (Heinze, Reiss, Rudolph-Albert, Herwartz-Emden, & Braun, 2009). Explanations for this relevance of language vary from the role of language for participating in classroom discourse (Civil, 2008), over the need to comprehend mathematical test items (Haag, Heppt, Stanat, Kuhl, & Pant, 2013) to the ability to use the language based thinking processes during learning (i.e. the epistemic function of language; Sfard, 2008). Researching these different explanations is important to understand language-related achievement differences and conceptualize instructional support to reduce them. In this contribution we focus on the first explanation: students' opportunities to actively participate in mathematical classroom communication are often put forward as a decisive reason for mathematical disadvantages in the literature on English Language Learners (e.g., Abedi & Herman, 2010). We propose a conceptualization of language-related Opportunities to Learn (OTL) with a specific focus on language-based learning processes. We propose an instrument to measure OTL in elementary mathematics classrooms. For this contribution we were also interested in relations between OTL and students' mathematics self-concept and achievement.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 115–122. Szeged, Hungary: PME.

THE CONCEPT OF OPPORTUNITIES TO LEARN MATHEMATICS

In the current literature on Opportunities to Learn, rather broad conceptualisations are used. OTL is defined, for example, as "what students learn in school related to what is taught in school" (Schmidt & Maier, 2009, p. 1). In that research, OTL is often measured by structural conditions of math instruction, for example the time spent in small-group instruction or the school's financial situation and equipment (Pianta, Belsky, Houts, & Morrison, 2007). Even though these conceptualizations surely describe relevant context conditions of mathematics instruction, they do not provide insights into the mathematical teaching and learning processes in the classroom.

From a mathematics education perspective, the communicative discourse between the individual students, their peers, and the teacher is one of the crucial determinants of mathematical learning processes (Sfard, 2008). Moreover, it is considered decisive that students can engage in authentic mathematical practices during learning (Hiebert & Grouws, 2007). To realize this, students are expected to communicate their thinking about mathematics in writing and orally (Civil, 2008; Gorgorió & Planas, 2001). As indicated above, it can be assumed that one reason for language-related disparities in mathematics learning lie in the participation in these communication processes. In particular, it is hypothesized repeatedly (Abedi & Herman, 2010; Civil & Planas, 2004; Heinze et al., 2009) that students with weak skills in the instructional language have fewer opportunities to participate in these discourse processes. Starting from this perspective, we define OTL in our research *as the opportunity to follow and participate in mathematical discourse processes in the classroom and to engage in mathematical practices*.

Opportunity to learn in this sense is strongly, but not solely dependent on the classroom environment provided by the teacher. For example, it can be assumed that students with low mathematics skills or low mathematical self-concept tend to refrain from active participation in a mathematical discourse, if the classroom atmosphere does not support them sufficiently (Liu & Wang, 2008). This would result in reduced OTL in the sense of the definition above. On the other hand, the provision of OTL is of course dependent on the practices of mathematics teacher (Baumert et al., 2010) to stimulate a rich discourse. Different dimensions of OTL can occur in a classroom discourse. In certain situations, a clear and structured verbal teacher instruction might offer substantial opportunities to construct initial mathematical knowledge (Hiebert & Grouws, 2007, Drollinger-Vetter, 2011) by decoding the information provided by the teacher and connecting it to prior knowledge. It is indisputable nevertheless, that sustainable learning also requires active mental processing of this information beyond taking up information (Baumert et al., 2010). Finally also practicing mathematical forms of discourse is put forward as an important preparation for own mathematical thinking (Sfard, 2008). In this vein, we distinguish between *receptive OTL* and *active OTL* in the sequel. While the first describes opportunities to access information offered by the teacher or peers in class, the second refers to opportunities for active elaboration of mathematical ideas and concepts.

Measuring Opportunities to Learn

To study the antecedents and effects of OTL, it is in many cases necessary to obtain valid and reliable measures of the opportunities students have and use for mathematics learning in the sense described above. In the past, different approaches have been developed to achieve this. The most valid paradigm to obtain information about opportunities to learn is surely the use of classroom observation or video analysis. For example, mathematical discourse processes have been analysed in regard to how transparent mathematical concepts are communicated in classroom discourse (e.g., Drollinger-Vetter, 2011) and related to student achievement. Nevertheless, students' individual thinking processes are not accessible in video recordings. Since a video-based approach is not feasible for practical reasons under some conditions, different and more distal indicators of OTL are often used. This comprises, for example, examining the tasks used in classroom instruction (Baumert et al., 2010), or student and teacher reports. Balancing validity and efficiency, well-constructed student self-reports have been used successfully in the past. For example, Abedi and Herman (2010) asked students about the content that was covered in their classrooms as a measure of OTL and found relations of this variable to the students' self-report on their understanding of their teachers' instructions, explanations and tests. These self-reported content coverage and comprehension ratings are inspired by more traditional definitions of OTL. In general, self-reports are frequently used to measure self-beliefs and were repeatedly shown to be valid instruments even in young age. For example, Ehm, Duzy, and Hasselhorn (2011) measured the mathematical self-concept already in grade 1 to study its relationship to mathematics skills and migration background.

Summarizing, even though more elaborate instruments for deeper analyses are desirable, efficient, reliable and valid self-report measures of OTL have a high potential for research on differences in individual learning processes in general, and language-related disparities specifically. Yet, validated instruments addressing students' opportunities to participate in classroom discourse are rare.

AIM AND RESEARCH QUESTIONS

The present study is part of the first author's PhD project that focuses on language-based explanations for migration-related disparities in mathematics learning. One goal was to develop and study a self-report instrument to measure language-related OTL in the elementary school mathematics classroom. The following questions guided our study: (1) Do students' self-reports reflect the theoretical differentiation of receptive and active OTL? (2) Do learners from German and non-German homes differ in their ratings of language-related OTL? (3) How are students' self-reports of language-related OTL connected to their mathematics skills and self-concept in both groups?

METHOD

In a cross-sectional survey design, data was collected from N = 383 German third graders (N = 163 students from non-German speaking homes) from 24 classrooms. A questionnaire with two subscales was developed, with items measuring receptive OTL

(ROTL) and active OTL (AOTL). Students were asked to give self-evaluations on their communicative (e.g. *"I often explain how to solve a task in math."*, 2 items) and cognitive (e.g. *"I often think about how to solve a task."*, 2 items) participation in classroom discourse as parts of the active OTL scale, as well as on their receptive OTL (e.g. *"I often only understand what to do in math, if our teacher explains the task several times."*, 4 items) on a four-point Likert scale (1: completely disagree to 4: completely agree). Mathematical self-concept was investigated with six items adapted from previous studies (Mullis, Martin, Ruddock, O'Sullivan, & Preuschoff, 2009). Mathematics skills were measured with 52 open items covering arithmetic skills (AS; e.g. 38 + 43), mathematical concepts (MC; e.g. finding the half of 56), word problems (WP) and understanding of manipulatives (UM; e.g. label numbers on the number line). Sum scores were computed from each half of the items of all arithmetic skill facets for further analysis (eight item parcels). Confirmatory factor analysis (CFA) and measurement invariance testing with *MPlus* (Muthén & Muthén, 1998-2015) was applied, using the WLSMV estimator for categorical data and robust standard errors to correct for the hierarchical structure of the data.

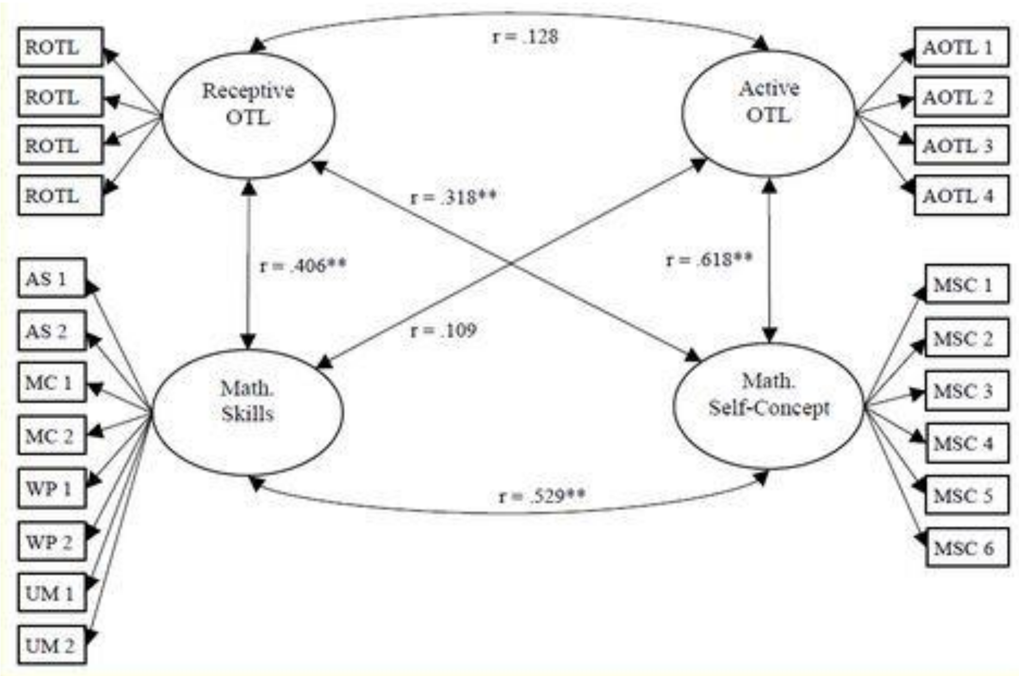

Figure 1: Full CFA model for receptive (ROTL) and active OTL (AOTL), mathematics skills and mathematical self-concept (MSC), **p < .01

RESULTS

The confirmatory factor analysis revealed a good fit for a measurement model of language-related OTL that differentiates the two subscales of ROTL and AOTL ($\chi^2 = 26.10$, $df = 19$, $\chi^2/df = 1.37$, RMSEA = .031). Also the full model including mathematics skills and mathematical self-concept and the OTL measures (Figure 1) showed good model fit ($\chi^2 = 252.99$, $df = 202$, $\chi^2/df = 1.25$, RMSEA = .026). The low correlation ($r = .128$, n.s.) between the two OTL facets supported the theoretical differentiation between the two OTL facets.

When considering the complete sample, receptive OTL is positively correlated with mathematics skills and mathematical self-concept. Active OTL only correlates significantly positively with mathematical self-concept and not with mathematics skills. Latent correlations of $r = .618$ (AOTL, $p < .01$) and $r = .318$ (ROTL, $p < .01$) indicated that the OTL constructs can be differentiated statistically from students' self-concept ratings.

In the next step, differences between learners from German and non-German speaking homes were analysed. A measurement invariance analysis using nested models indicated full scalar invariance between students from German and non-German speaking homes. This ensures that means and correlations from the analysed model (Figure 1) can be compared meaningfully between the two groups.

Comparing children from German and non-German speaking homes in analyses of variance (ANOVA, Table 1) revealed differences in self-reports between students from non-German and German speaking homes with small effect sizes. Students from non-German speaking homes reported more active OTL, less receptive OTL but equal self-concept as students from German-speaking homes. The expected large difference in mathematics skills was observed, as well.

	Non-German		German			
	M	*SD*	***M***	*SD*	***F***	***η^2***
Receptive OTL	**2.72**	0.78	**2.95**	0.73	8.90**	**.023**
Active OTL	**3.01**	0.54	**2.86**	0.57	6.66*	**.017**
Mathematical Self-Concept	**3.23**	0.70	**3.28**	0.69	0.46	**.001**
Mathematics Skills	**51%**	18%	**63%**	17%	44.31**	**.104**

Table 1: Means and standard deviations (by the language spoken at home), statistics from group comparisons, *$p < .05$, **$p < .01$

A comparison of correlations among latent variables in our CFA model (Figure 1) showed differences between the two groups (Table 2). While the two subscales of language-related OTL can be clearly separated for learners from non-German speaking homes ($r = .035$, n.s.), they are significantly, but weakly correlated for learners from German speaking homes ($r = .258$, $p < .01$). Moreover, a higher amount of active OTL in learners from non-German speaking homes goes along with higher mathematics skills, while this relation is substantially weaker for students from German speaking homes. Receptive OTL is related to mathematics skills in both samples substantially.

Variable		1	2	3	4
1. Receptive OTL	German	–			
	Non-German				
2. Active OTL	German	.258**	–		
	Non-German	*.035*			
3. Mathematics Skills	German	.388**	.137	–	
	Non-German	*.420***	*.296**		
4. Mathematical Self-Concept	German	.389**	.557**	.495**	–
	Non-German	*.246***	*.769***	*.688***	

Table 2: Correlations of the latent means for children from German (N = 163) and non-German (N = 220) speaking homes, *p < .05, **p < .01

DISCUSSION

In this contribution, an instrument to measure language-related OTL in the mathematics classroom was presented, evaluated and studied for similarities and differences between learners from German and non-German speaking homes. A good CFA model fit indicated that the developed items are sufficiently reliable to survey language-based OTL and measurement invariance analyses showed no differences in the internal structure of the single scales between the two groups. This means that our instrument can be used to compare learners from German and non-German speaking homes in their self-reported OTL, ensuring that the same construct is measured in each group.

Compared to learners from German speaking homes, students from non-German speaking homes reported less classroom discourse understanding (ROTL), slightly more active participation in classroom discourse (AOTL) and equal self-concept than those from German-speaking homes. While this resembles the expected pattern for receptive OTL, it was unexpected for active OTL. Since both groups showed comparable self-concept ratings, an over-estimation of active OTL due to higher self-concept is not a plausible reason. A different explanation might be that students from non-German speaking homes compensate their lower mathematics skills and classroom discourse understanding by engaging more (or perceive to engage more) in the classroom interaction.

As expected theoretically, opportunities to follow classroom instruction (ROTL) and active participation in class (AOTL) are positively related to mathematics skills and students' self-concept. While these relations are mostly comparable for both language groups, the relation between active OTL and mathematics skills goes back primarily to students from non-German speaking homes. While the overall result provides additional support for the relevance of receptive and active participation in the instructional discourse (Civil & Planas, 2004; Drollinger-Vetter, 2011) in general, it was unexpected that the relation between AOTL and mathematics skills was much weaker for students from German-speaking homes. Several explanations for this result are possible. It would be problematic, if this relation would be primarily due to confoundations of AOTL with self-concept in the non-German sample. This warrants further research. On the other hand, the importance of active participation in classroom discourse lies in the language-based elaboration of mathematical knowledge. This process might be less relevant for the development of students' mathematics skills, if these students already possess a certain level of skills in the language of instruction, which allows them to engage in these elaborations without frequent active participation in the classroom discourse. This would result in a weaker relation for students from German-speaking homes. We would not expect such a result for measures that capture purely mental cognitive elaboration of mathematical ideas (Baumert et al., 2010), but for our measures that covers communication-related aspects of classroom discourse this could explain our finding.

Our study was – to our knowledge – the first to differentiate between receptive and active OTL. Before strong conclusions can be drawn, of course further research is needed. In particular our result, that active participation in the classroom discourse seems to be particularly important for students who are less familiar with the language of instruction, must be considered preliminary in this sense, even though it is in line with opinions in the literature (Civil & Planas, 2004). Nevertheless, if this result is substantiated with other methods and more elaborate research designs, this would offer empirical evidence that actively engaging students with low language skills in the classroom discourse is at least equally important as supporting them to follow the instructional discourse (for example by using language accommodations).

To summarize, our instrument proposed a first effective approach to measure language-related OTL using self-reports. Our first results indicate that this instrument can be used for analysing differences and similarities between learners from German and non-German speaking homes at least for receptive OTL. The active OTL measure requires further analyses: The effects found here can be explained, but were unexpected based on our theoretical assumptions. Moreover, the cross-sectional design of this first study can only provide preliminary insights and does not support strong causal claims. Validation studies with other instruments, longitudinal and intervention studies are necessary to shed more light on the role of OTL for mathematics learning in students from different language or migration backgrounds.

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PROSPECTIVE HIGH SCHOOL TEACHERS' PROBLEM SOLVING ACTIVITIES THAT FOSTER THE USE OF DYNAMIC GEOMETRY SOFTWARE

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In this research report we analize and discuss the extent to which prospective high school teachers engage in problem solving experiences that involve the formulation of conjectures and relationships, ways to test and validate those conjectures and the extension or generalization of results. Ten prospective high school teachers participated in a course that promoted the use of GeoGebra in a problem solving environment during one semester. Results indicate the importance for teachers of developing ways to appropriate a set of technology affordances to represent and explore mathematical tasks. In this process, they recognized that the construction of dynamic models of tasks not only helped them identify mathematical relations, but also ways to validate or support them.

INTRODUCTION

The experience developed over the years and the absence of technology in the secondary school classrooms lead us to rethinking the validity of the teaching methodology followed in our teacher training courses. We also ask why the prospective teachers, who manage the software skilfully, do not use and do not take advantage of the potential of the Dynamic Geometry Software (DGS) for planning mathematics lessons. The teaching process followed during the training period of the prospective secondary mathematics teachers was guided by two ideas: on the one hand, the idea that mastering software techniques was enough to be a competent mathematics teacher and confident to use technology for learning and teaching mathematics, and on the other hand, the importance of the type of mathematical tasks for unpacking the potential of technology in the learning of mathematics. We have found that instrumentalization and technical training is not enough to appropriate the tool, beyond its use to illustrate concepts. Several studies (Santos-Trigo & Camacho, 2009, Santos-Trigo, Camacho-Machín & Moreno-Moreno, 2013 and Santos-Trigo, Camacho-Machín & Olvera-Martínez, 2014) stress the need to provide prospective mathematics teachers with opportunities for using the dynamic software to enhance the mathematical activity that arises from problem solving environments.

This work is part of a research project: "Problem Solving and Technology for Professional development of the secondary mathematics teachers", whose aim is to characterize the knowledge that prospective secondary mathematics teachers need for developing problem solving activities in classroom, using computational tools and providing a guide work which allows them to structure and organize the teaching of

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 123–130. Szeged, Hungary: PME.

mathematical content based on the use of technological tools in problem solving environments (Moreno, Camacho & Azcárate, 2012).

Therefore, the specific objectives of this research report are:

- To characterize the mathematical activity of the students in the context of problem solving using DGS and "paper and pencil"
- To develop a protocol to guide the resolution of tasks and which enhances the appropriation of DGS as a resource in teaching mathematics.

CONCEPTUAL FRAMEWORK

The mathematical knowledge of secondary school teachers plays an important role in teaching the discipline, however, it is necessary to investigate the effects of that knowledge domain of teachers in the teaching and learning scenarios. This mastery of mathematical knowledge is not enough to promote a conceptual understanding in their future secondary school students based on their teaching practice. The term *Mathematical Knowledge for Teaching (MKT)* is usually used to refer to what teachers need to know in terms of content, representations, ways of thinking, how to act and what to use so that their students develop strong and solid mathematical knowledge. This idea, backed by various studies (Ball, Thames & Phelps, 2008, Davis & Simmt, 2006), suggests some relevant research questions on the relationship between the mathematical knowledge of teachers and the quality of teaching: How is this knowledge expressed in teaching practice? For example, do the most proficient teachers, in terms of mathematical knowledge, offer different ways and strategies of teaching and focus their attention towards their students' construction of meanings and connections of concepts? Do they only show fewer mathematical errors than their colleagues with less mastery of the discipline or topic under study? What distinguishes a teacher with solid mathematical knowledge for teaching in their classroom practices? How does a deficiency of mathematical knowledge for teaching restrict the development of activities in the classroom? etc.. Regarding to Conner, Wilson & Kim (2011), mathematical competence or proficiency for teaching secondary school teachers is structured around three dimensions or categories that are not independent, but interrelated components characterizing the mathematical knowledge of teachers for teaching:

- Mathematical Ability (aspects of mathematical knowledge and skills).
- Mathematical Activity (process to do mathematics).
- Mathematical work of teaching (skills that enable teachers to integrate their knowledge and the processes to increase the understanding of their students).

Another increasingly important issue for teachers is the need to incorporate the systematic use of various technological tools into their mathematics teaching and learning scenarios to help secondary school students develop an understanding of mathematics based on their competence to solve problems.

The present research involved the collaboration of several researchers from different universities where secondary school teachers are being trained, with the aim of developing and evaluating the design of a set of tasks in the terms mentioned in the previous paragraph. We look at the mathematical activity shown by prospective mathematics teachers in solving an optimization problem with “pencil and paper”, and we compare it with the mathematical activity that emerges when DGS is used to solve the problem.

METHODOLOGY

The research work was performed with ten prospective high school teachers in the third year of the mathematics degree. As researchers, we selected several tasks from baccalaureate or similar textbooks and analyse and discuss the potential of these tasks through different ways of resolution and environments (Santos-Trigo & Camacho-Machín, 2009).

The experimental work was divided into three phases. Although prospective mathematics teachers were engaged with three tasks, in this research report we focus on only one of them. In the first stage, prospective mathematics teachers solve tasks individually, using “paper and pencil”. The second phase was to develop a training workshop with the DGS, which consisted of eight sessions of one and half hour each, so that the prospective mathematics teachers acquired a mastery of instrumentalization techniques (Trouche, 2005) for their own use. Finally, in the third phase, they worked in pairs to solve the same three tasks, but now, using the GeoGebra software. The following three questions were asked to the prospective mathematics teachers to help them to reflect on the mathematical activity and on the use of DGS:

- What other ways of solving each of the tasks does GeoGebra suggest to you?
- Can you propose a variant of the initial one which allows an extension of the task? (Santos-Trigo & Camacho-Machín, 2009)
- What is the new mathematical knowledge involved in the solution which has not been taken into account in the solution performed with “paper and pencil”?

Through an inductive process, coming from the analysis of the first phase (individual resolution using “paper and pencil”) and from the third phase (solutions which were worked out by the five pairs of students using GeoGebra), we have identified some features of the mathematical activity developed by the students, and we have analysed the role of DGS for learning mathematics.

The analized task is as follows and a detailed analysis of the potentialities of the task can be seen in (Santos-Trigo, Camacho-Machín & Olvera-Martínez, 2014):

ABCD is a rectangle, AB has a length of 6.5 cm, BC has a length of 4 cm. M is a point on segment AB. N is a point on segment BC, P is a point on segment CD, and Q is a point on segment DA. It is known that AM=BN=CP=DQ. Where should point Mbe located in order to minimize the area of the quadrilateral MNPQ?” Could you solve the task in different ways depending on the course in which you could propose the task?

DISCUSSION OF THE RESULTS

Phase 1 ("pencil and paper"). The two main strategies that prospective mathematics teachers used to solve the task were substractive and additive strategies: -Subtractive strategy (Figure 1): minimizing the area of the quadrilateral

-Additive strategy: maximizing the area of the inner triangles, as is shown in the solution (Fig. 2).

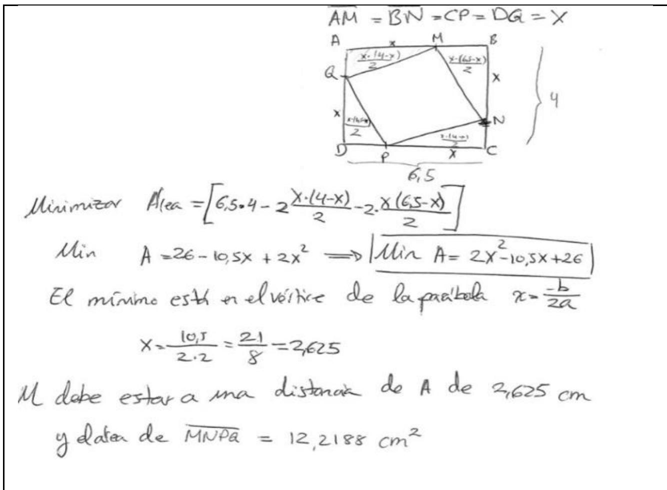

Figure 1. Substractive strategy

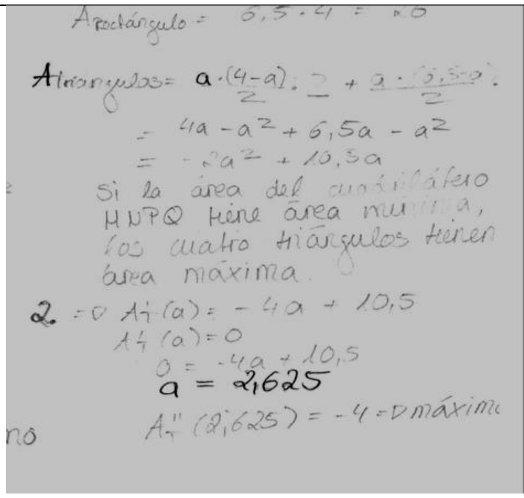

Figure 2. Additive strategy

One prospective teacher uses a Cartesian approach (Figure 3) to define the vertices of the rectangle and justifies that the inscribed quadrilateral is a parallelogram with its sides as vectors verifying the condition of parallelism, then, he calculates the area of the triangles and maximizes it. He adds the fact of calculating the area of a parallelogram as another possibility, but he states that "it would be more complicated as it is a rectangle, moreover, the position of the figure MNPQ would complicate the calculations "(Fig. 4).

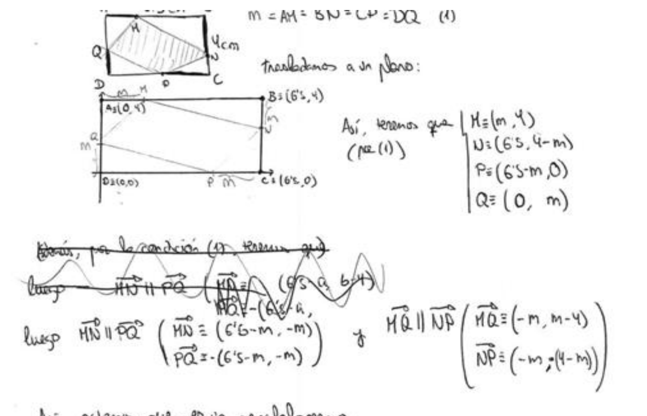

Figure 3 . Cartesian approach

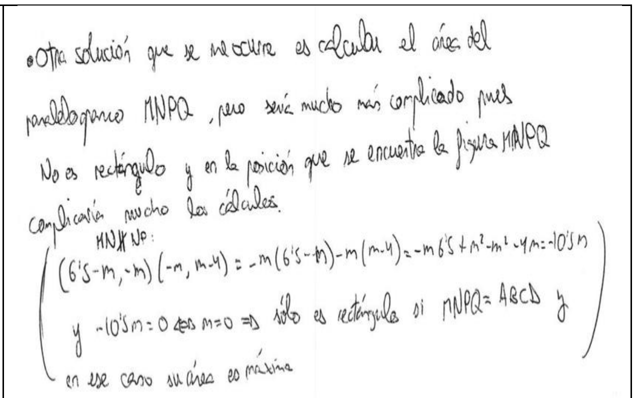

Figure 4. Parallelogram area

For minimizing the area, several of the prospective mathematics teachers propose finding the first derivative and setting it equal to zero, although there is an absence of checking and justification with the second derivative test. They accept that the obtained value corresponds to a minimum or recognize that the function to minimize is a parabola and the minimum corresponds with the vertex of the parabola (*"x = -b/2 a*")

We observe a lack of rigour in the responses of the prospective mathematics teachers. They do not use a suitable algebraic notation, in spite of the fact that the algebraic calculation is the one preferred by almost all prospective mathematics teachers. It is also noteworthy that there is an absence of adequate justifications or mathematical proofs. Now, we go on to analyse the work the students did with DGS.

Phase 3 (GeoGebra in pairs): three pairs "copied" the solution which was individually developed in Phase 1, they get more precise images by using the software. There are certain differences in how each pair uses the software and, therefore, in the role it plays. Some interesting solutions are shown below.

One pair is able to connect the following three representation modes: geometric, numeric and graphic, and they check the same solution obtained in each of the systems of representation (Figure 5). And another couple only connects the geometric and the analytical modes of representation. They use the GeoGebra tool called "Function study" to provide the necessary information, once the function is defined as area (Fig 6). They do not use the numerical representation mode. Another couple understand that GeoGebra enables them to graphically represent the parallelogram area function which helps them to visually check the result obtained in Phase 1. They believe that the said software aids the graphic demonstration of a wide range of problems when it is graphically represented after obtaining the function to be optimized.

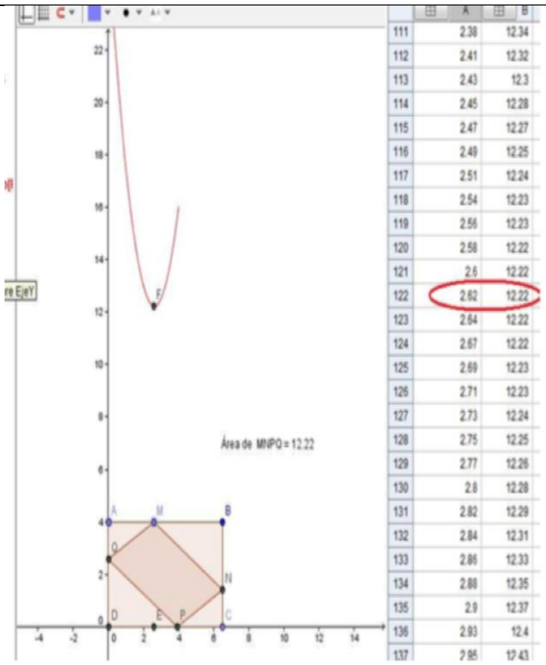

Figure 5. Three representation systems

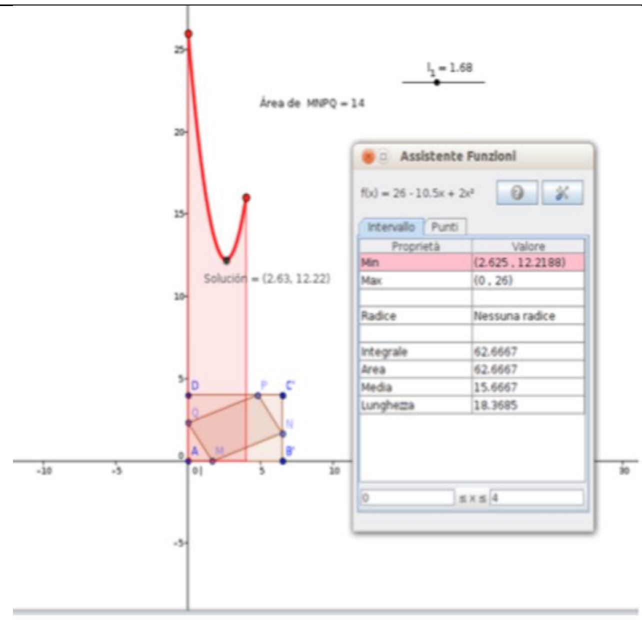

Figure 4. Function study

Regarding the extensions of the problem, most students do not make many significant contributions. On the one hand, they propose the extension in terms of changing the rectangle for regular polygons, in their view, the minimum area of the inscribed polygon is obtained when the vertex is at the midpoint of one of the sides. On the other hand, they propose minimizing the perimeter of the inscribed figure as an extension activity and anticipate a possible solution.

To summarize, as regards the mathematical knowledge involved in the solution using GeoGebra, the students are not aware of the new mathematical knowledge arising from a dynamic approach to the task. For example, some students do not consider the use of the three systems of representation provided by GeoGebra as new knowledge. Neither do consider the importance of mastering the existence of the function as being new knowledge, which appears to be implicit in the decision to move the point on the shortest side of the rectangle so as not to lose the effects of "drag" from the parallelogram. This absence of reflections on the mathematical knowledge, which arises in solving the tasks, requires asking questions that help them to make such knowledge explicit.

SOME FINAL REMARKS

In this research report we intended to characterize the mathematical activity, in terms of Conner et al. (2011), of ten mathematics students' involved in a course of training secondary mathematics teachers when they might solve the same problems by hand ("paper and pencil") or using DGS. The results have not met our expectations because the students were not able to explore and experiment the potential of the dynamism provided by GeoGebra. As teacher trainers, we believe it is important to provide our students with more specific information to facilitate the appropriation of the tool as an instrument to develop mathematical activity and develop problem solving skills. We have found that being able to manage the tool skilfully is not enough to take full advantage of it as a knowledge generator. Students have difficulties in making the mathematical activity (defining, justifying, arguing, testing, generalizing, etc.) explicit performed during the resolution of the tasks with "pen and paper" and with the DGS. For them, there were almost no differences between both ways to solve the tasks. Students were not able to communicate the cognitive processes that emerge from the task, neither identify key moments of the resolution nor procedural development, which is the first step for planning a lesson, guiding the learning of the students, anticipating responses, etc. It also seems essential to reflect about the pertinence of mathematical formalism that should be required depending on teaching situations. We have prepared a guide (implementation guidelines), for solving tasks using dynamic software as a learning tool. The guide intends to help prospective mathematics teachers to make the mathematical activity that emerges during the process of resolution explicit:

A dynamic approach using GeoGebra: *Comprehension***-** Identify the elements that make sense of the problem. *Analyzing*- analyse from a geometric perspective.

Communicating- comment on the process. *Exploration*- What properties can be observed from the movement of the set of generated figures? Are the basic characteristics giving rise to the constructed geometric model maintained? Find the variation domain of the dynamic model. Obtain a Cartesian representation linking what you want to optimize (dependent variable) with a value that can be considered as an independent variable and find the geometric locus which is obtained. Could you use more than one independent variable? Is there a pattern associated with the chosen variables? Visually identify at what point the graph obtained reaches the maximum / minimum value. Build (using Excel) a table showing some values of the variables that model the problem.

An Algebraic approach: *Communicating*- Choose an appropriate notation and find an algebraic expression for the function to be optimized. Obtain the dependence of variables and relationships for converting the expression into a real function of a real variable. Find an algebraic expression for the function to be optimized. *Conceptualization*- Graphically represent the function obtained and discuss the properties. *Connecting*- What is the relationship between the obtained function and the functional model obtained from the dynamic approach? Apply the derivative test to obtain optimum points to find the value(s) that optimizes or optimize the function obtained. Compare this value with the previously obtained value by using the software. Could a more general case be found?

Extensions: Modify some of the initial conditions and try to extend the original problem.

Acknowledgments: This research report has been financed by the project EDU2011-29328. and EDU2014-54526-R of the National Plan of the Spanish Ministry of Education and Science.

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FUNCTIONAL RELATIONSHIPS IDENTIFIED BY FIRST GRADERS

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This study forms part of a broader research project conducted in Spain on functional thinking as an approximation for algebraic thinking in elementary school students (ages 6-12). The two types of functional relationships identified by first year (6-year-old) pupils are discussed in this article. We present results from two sessions of a teaching experiment developed. Most students revealed the ability to detect correspondence or co-variation relationships. Most of the former were inappropriate and the latter appropriate.

INTRODUCTION

This study lies within the realm of the early algebra proposal, according to which algebraic thinking should be introduced in mathematics from the earliest years of schooling (Kaput, 1998) as a way of precluding secondary school students' difficulties with algebra. Some of these difficulties revolve around students' inability to understand the relationship between two data sets (MacGregor & Stacey, 1995; Warren, 2000). This understanding is one of the keys to a functions-based approach in early algebra. As Schliemann, Carraher, and Brizuela (2012) observed, "a functions-based approach to early algebra relies on the importance accorded to sets of values and to ordered pairs from a domain and target" (p. 110).

Research on students' functional thinking shows that this approach should be introduced in the early years of schooling and gradually developed, for functions are not readily understood (Chazan, 1996). A longitudinal study by Brizuela and Martínez (2012) confirmed that early experience with tasks entailing functional relationships is beneficial in the long run. Traditional assumptions about elementary school pupils' limited ability to work with functional elements have been challenged by prior studies (Blanton & Kaput, 2011; Brizuela & Martínez, 2012). In recent decades, early algebra researchers have shown that children acquire functional thinking-related aptitudes at younger ages than believed (e.g., Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, 2015). Cañadas, Brizuela, and Blanton (2016) recently provided a detailed description of these relationships in the context of functional thinking for second year elementary school students. That study portrayed students' understanding of functional relationships and the differences in the ways they expressed it.

The results of the aforementioned studies showed that elementary education students can engage in tasks involving functional thinking. Moreover, the authors generally agree that very young (particularly pre-school and first-year elementary school) students' functional thinking should be explored in greater depth (Cañadas and Molina, in press).

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 131–138. Szeged, Hungary: PME.

While research has shown that introducing functional thinking is beneficial for elementary school children, very few studies have addressed the subject in Spain (Cañadas & Fuentes, 2015). The possible implications for classroom practice in the wake of the recent inclusion of functional relationships in the Spanish curriculum (Ministerio de Educación, Cultura y Deporte, 2014) lend the subject particular relevance at this time.

This paper on functional thinking in first year elementary school students focuses on the functional relationships exhibited by the Spanish children involved.

FUNCTIONAL THINKING AND RELATIONSHIPS

Functional thinking, one of the approaches deployed in early algebra, centres on the processes that come into play when working with functions, i.e., quantities exhibiting co-variance (Cañadas et al., 2016). Functional thinking is based on the construction, description and reasoning, with and about functions (Cañadas & Molina, in press).

Consequently, functions are the mathematical content involved in functional thinking. Using linear functions, early elementary schools pupils can be confronted with situations that afford them the opportunity to identify relationships. The symbolic algebraic expression of these functions is $f(x)=ax+b$, where a and b are natural numbers. Representational systems other than algebraic symbolism, such as verbal, manipulative or tabular expression, play a prevalent role in the early years.

In functions, variables indicate a quantity that in a given numerical set may adopt different values depending on the nature of the problem. In the equation $y=f(x)$, x represents the independent and y the dependent variable (because its value depends on the value of x). Correspondence is the relationship established between each pair of values $(x, f(x))$.

In the correspondence relationship, the focus is on the relationship between two sets and on explicitly stating an (algebraic) rule (Confrey and Smith, 1995). In contrast, in a co-variational approach, the links between domain and range are spatial and relational, and the (algebraic) rule is only “a derived characteristic” (p. 79). In the co-variational approach, linear functions have a constant first difference.

Co-variation implies correspondence because the variation in the values of one variable with respect to the other is attendant upon the pairs of values $(x, f(x))$.

Based on prior research, Smith (2008) identified three types of approaches to working with functional thinking in the earliest years of schooling: (a) recursive patterning; (b) co-variational thinking; and (c) correspondence relationships. These three approaches, associated with the functional relationships stemming from the mathematical study of functions, are described below.

Recurrence or recursive patterning involves identifying the pattern of variation in a series of values and hence entails a single variable.

In *correspondence* relationships two variables are found to be correlated. The focus is on the relationship between two sets, i.e., on identifying the relationship between the values of two sets of data and establishing the general rule or pattern governing that relationship.

Co-variational thinking is based on the analysis of how two quantities vary simultaneously and how change in the values of one variable induces change in the values of the other.

RESEARCH OBJECTIVE

This study describes the functional relationships identified by Spanish first year elementary school pupils when solving problems designed to further functional thinking.

METHOD

The findings discussed hereunder derive from a study conducted on an intentional sample of 30 first year (6-7-year-old) elementary school pupils enrolled in Granada, Spain. The reasons for choosing the school were the institution's and its teachers' willingness to participate and its geographic location within the region where the broader research project is underway. The pupils had not previously been exposed to problems involving functional thinking (nor was this content on the Spanish curriculum at the time).

The teaching experiment designed and conducted consisted in five approximately 90 minute classroom sessions. Three researchers *in situ* collected the data, one of whom assumed the role of teacher-researcher.

This paper focuses on sessions 2 and 3, both of which were video-recorded. The setting for the task performed in these sessions was a kennel in which each dog needed its own feeding bowl and which had an additional five bowls to be shared for water. The functional relationship, then, was of the type $y=x+5$. Pictures of dogs and bowls were used and the situation was described verbally in the interaction with the whole group in the classroom. The students were asked by near and far particular cases. Then they had time to work individually on a questionnaire. After which their solutions were discussed aloud, in the whole classroom. The questions asked followed the inductive reasoning model proposed by Cañadas and Castro (2007). The videos were fully transcribed. We used two videcameras in the classroom: The first one for recording the whole classroom; and the second for the specific moments during the individual work.

DATA ANALYSIS AND RESULTS

The data analysis drew an initial distinction between responses that did and did not evince functional relationships. The answers that entailed such relationships were then grouped by type (recurrence, correspondence or co-variation). Relationships were also classified as appropriate to the problem or otherwise.

We identified students exhibiting functional relationships from those who did not. The type of relationships established by the former was defined. Each pupil was identified with the letter S and a number from 1 to 30. Students who reached at least one inappropriate conclusion were labelled with an asterisk (*).

- Sin evidencia de relación funcional: S1-S4-S12-S15-S17-S23
- Recurrencia: -
- Correspondencia:S2-S2(*)-S3(*)-S5-S6-S6(*)-S7-S7(*)-S9-S10-S12(*)-S13-S13(*)-S15(*)-S16-S17-S17(*)-S18-S19-S20-S21-S21(*)-S26-S26(*)-S28-S28(*)-S29-S29(*)-S30-S30(*)
- Covariación: S7-S12-S12(*)-S13(*)-S17-S21-S25-S28

All six students who showed no sign of having identified functional relationships answered the problem correctly.

No recurrent relationships were observed in any of the students' replies.

Correspondence was the relationship most frequently exhibited by the students (20). The relationships established by 13 of these 20 pupils were inappropriate for the problem.

Seven of the students who detected a correspondence relationship also identified co-variation between the variables involved. Two of these students established an inappropriate relationship.

For reasons of space, the above summary of the students' replies cannot be analysed in detail here. One student (S17) answered differently depending on the functional relationship considered. Fragments of a conversation with him are transcribed below. Although initially his reply displayed no functional relationship whatsoever, as the experiment progressed he was observed to draw both correspondence and co-variation relationships

Fragment 1. S17 draws no functional relationship

In the following fragment S17 replied to the teacher-researcher's (I1) question about the total number of bowls needed for three dogs.

1. I1: […] yes, S17?
2. S17: Eight.
3. I1: Eight. Why?
4. S17: Because there are three (pointing to the three dogs with three feeding bowls in Figure 1) and five more (pointing to the five water bowls in the figure) makes eight.

To find the answer, S17 observed the manipulative material furnished by I1 when introducing the problem, which showed three feeding and five water bowls (Figure 1). He consequently added 3+5 to find the answer to the question. He found eight bowls for three dogs, but without perceiving any relationship between the number of bowls and the number of dogs.

Figure 1. Dogs, feeding bowls and water bowls

Fragment 2. S17 draws a correspondence relationship

At the beginning of session 3, I1 asked the students whether they remembered what they did in session 2. In the following fragment, S17 explained the task of session 2 with an example, proposed by the students himself, in which he established a correspondence relationship.

24. S17: We had some dogs. Then we had to give them doggy bowls from the bowls we had... For instance, there were ten, we had to add five.

25. I1: Why?

26. S17: Because the dogs only had five water bowls.

In the example proposed, to find the total number of bowls S17 added the feeding bowls (which he equated to the number of dogs) to the number of water bowls. The student identified the five water bowls as a fixed amount (line 26) and explained that it should be added to the number of feeding bowls (one per dog, line 24), to thereby find the total number of bowls needed for 10 dogs.

Fragment 3. S17 draws a co-variation relationship

After working with situations involving one, two and three dogs, I1 asked how many bowls would be needed for five, positioning two more dogs with their respective feeding bowls in the window. The following fragment shows that S12 and S17 identified a co-variation relationship.

I1:[…] Now a tougher question. Suppose more dogs come […] to the kennel. Look what we're going to do now. This one comes and we give it its food and then this other one comes and we give it its food too (I1 pastes two more drawings of dogs and their feeding bowls alongside the initial three), OK? How many bowls do we need now?

[…]

We have five dogs, and how many bowls?

S12: Ten.

I1: How do you know that, S12?

S12: There were eight and if you added two more that makes ten.

[…]

I1: OK, S17, do you see it that way or some other way?

S17: I see it like S12. If we had eight and you added two more, we'll have ten.

In this fragment the students realised that if the kennel admitted two dogs, it would need two more bowls. In this case, both students noticed how one variable varies (the quantity of bowls) attending to how the other variable (the quantity of dogs) changes.

CONCLUSIONS

Despite their unfamiliarity with problems designed to further functional thinking, most of the students were able to detect functional relationships based on the specific situations described to them. Some replies showed no signs of functional relationships, although none were necessary to answer the questions, as attested to by the fact that all the students who established no relationships came up with the right answer. Although recurrence is considered as the most basic of the three relationships addressed here, none of the students identified it. No evidence of functional relationships was observed in students' replies to specific near questions. As the process of inductive reasoning progressed toward generalisation, correspondence and co-variational relationships were normally observed more frequently. Functional relationships arose in students' replies as the numbers involved in the problems grew.

Although correspondence was the relationship most frequently identified by students, the number detecting this type of relationship inappropriately might infer that they considered it locally for the specific questions posed. When larger numbers were involved, however, co-variation and the number of appropriate relationships rose.

Some students verbally generalised the relationship. Further to the above fragments of conversations with student S17, whose academic performance was average for the class, he generalised the correspondence relationship. In fragment 2 he resorted to an example with 10 dogs but, in light of his reply, he might be expected to follow the same process in other specific cases. In the student presented, as in another student, we observe that they select particular cases but the procedures and reasoning are valid for more cases that are part of the same clase. These particular cases can be considered generic examples in terms of Balacheff (2000).

Cañadas et al. (2016) noted that Spanish students tend to generalise functional relationships as correspondence rules. Nonetheless, in contrast to that study, the Spanish students in the present sample showed no signs of recursive patterning. These differences may be attributable to the functional relationship involved in the problem posed and the systems of representation used for the tasks.

Acknowledgment

The research reported here was developed within the Spanish project of Research and Development with reference code EDU2013-41632-P, financed by the Spanish Ministry of Economy and Competitiveness.

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AN EXEMPLARY MATHEMATICS TEACHER'S WAYS OF HOLDING PROBLEM-SOLVING KNOWLEDGE FOR TEACHING

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This case study investigated the mathematics problem-solving knowledge for teaching [MPSKT] of an exemplary secondary mathematics teacher. Particular focus was on how the knowledge was held to make it usable for effective teaching of problem solving. The study was framed in a perspective of MPSKT consisting of 5 components of knowledge. Analysis of data that included extensive interviews and classroom observations indicated that there were important relationships in how the teacher held her knowledge, with a core or anchoring component of knowledge giving meaning to the other components of knowledge in MPSKT and was critical to how she taught PS.

INTRODUCTION AND LITERATURE REVIEW

> "The teaching of problem-solving in arithmetic offers one of the greatest challenges to elementary-school teachers" (Johnson, 1944, p. 396).

> "One does not have to look far to establish the fact that success in teaching problem solving procedures is very limited" (Earp, 1967, p. 182).

Mathematical problem solving [PS] continues to be a challenge for many teachers to teach effectively and students to learn proficiently. While both student- and teacher-related factors have been identified as contributing to this, recent focus on mathematics knowledge for teaching suggests the importance of understanding mathematical problem-solving knowledge for teaching [MPSKT] in order to support teachers' change/growth. This paper contributes to this by reporting on a study that investigated the MPSKT of an exemplary secondary mathematics teacher. In particular, the study focused on how the knowledge was held to make it usable for effective teaching of PS.

PS has a special importance in mathematics education (NCTM, 2000) and thus should be a central aspect of teachers' knowledge for teaching mathematics. However, studies have highlighted issues with, in particular, prospective teachers' PS ability and knowledge of PS. For example, they tend to lack flexibility in choice of PS approaches (van Dooren, Verschaffel, & Onghena, 2003), apply a stereotypical solution to a problem (Leikin, 2003), and make sense of PS as a linear process (Chapman, 2005).

In order to address these and other concerns, both practising and prospective teachers have received attention in studies that investigated ways of supporting their learning and teaching of PS. For example, they have been engaged in: PS using a variety of strategies (Szydlik, Szydlik & Benson (2003), multiple-solution tasks (Guberman & Leikin, 2013), tasks with potential to promote creativity in PS (Levenson, 2013), the role of facilitating students' mathematical PS (Lee, 2005), and pedagogical skills in navigating PS and listening to students (Leiken, 2003).

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 139–146. Szeged, Hungary: PME.

Such studies on teachers provide insights for improving specific aspects of their MPSKT. But they do not address the combination of knowledge teachers should have and how they ought to hold it. Teachers' knowledge of and for teaching for *PS proficiency* (Chapman, 2014) must be broader than knowing how to solve problems and include a deep understanding of other factors that are associated with the development of proficiency in PS. This study examined the nature of this knowledge as held by an exemplary teacher.

THEORETICAL PERSPECTIVE

Genuine PS is considered here as "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 52); "finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable" (Polya, 1962, p. v). Adopting Ball, Thames, and Phelps's (2008) perspective that general mathematical ability does not fully account for the knowledge and skills needed for effective mathematics teaching, this study assumes that the knowledge needed to effectively teach genuine PS should be more than general PS ability. Teachers need to hold knowledge of PS for themselves as problem solvers and to help students to become better problem solvers. This is supported by works such as Mason, Burton and Stacey (2010), Mayer and Wittrock (2006), Polya (1962) and Schoenfeld (1985) and literature on PS that suggests what ought to be included in this knowledge, which provides the theoretical perspective of MPSKT used in this study.

Based on a review of literature on PS in mathematics education from 1922 – 2013, Chapman (2014) proposed a perspective of MPSKT for PS proficiency. Table 1 provides key components of it, which could be classified as: (1) PS content knowledge consisting of knowledge of problems, PS, and problem posing; (2) Pedagogical PS knowledge consisting of knowledge of students as problem solvers, and instructional practices for PS; and (3) Affective factors and (cognitive) beliefs. Chapman explained that this category-based perspective does not provide a complete picture of MPSKT since it does not account for relationships among the components. Understanding possible interdependence of them could be important to help teachers to hold MPSKT so that it is usable in a meaningful and effective way in supporting PS proficiency in their teaching. This study contributes to this by investigating this knowledge from the perspective of an exemplary teacher in order to determine a practice-based orientation of it regarding possible relationships among the components.

Knowledge of:	Description
Mathematical PS proficiency	Understanding what is needed for successful mathematical PS
Mathematical problems	Understanding of the nature of meaningful problems; structure and purpose of different types of problems; impact of problem characteristics on learners
Mathematical PS	Being proficient in PS Understanding of mathematical PS as a way of thinking; PS models and the meaning and use of heuristics; how to interpret students' unusual solutions; and implications of students' different approaches
Problem posing	Understanding of problem posing before, during and after PS
Students as mathematical problem solvers	Understanding what a student knows, can do, and is disposed to do (e.g., students' difficulties with PS; characteristics of good problem solvers; students' PS thinking)
Instructional practices for PS	Understanding how and what it means to help students to become better problem solvers (e.g., instructional techniques for heuristics/strategies, metacognition, use of technology, and assessment of students' PS progress; when and how to intervene during students' PS).
Affective factors and beliefs	Understanding nature and impact of productive and unproductive affective factors and beliefs on learning and teaching PS and teaching

Table 1: Components of MPSKT (Chapman, 2014, p. 22)

RESEARCH PROCESS

This study is part of a 4-year national funded project that investigated elementary and secondary mathematics teachers' thinking and teaching of PS using contextual/word problems. The **methodology** for it is case study (Stake, 1995) to allow for an intensive investigation, framed in a naturalistic research perspective that focuses on capturing and interpreting peoples' thinking and actions based on actual settings through an emergent approach (Corbin & Strauss, 2008). The **participant** was a high school math teacher (pseudonym, Cintia) for 16 years. She received a national teaching award, provincial teaching awards and other awards as an outstanding/exemplary math teacher and was co-author of secondary mathematics textbooks used in the province. Her teaching was inquiry based with emphasis on students' understanding of math.

Main **sources of data** were open-ended interviews, PS tasks, classroom observations, role play, teaching/learning artefacts, and students' work. As part of the larger project, the interviews explored the participant's knowledge (understandings/conceptions) of and experiences with PS in three contexts: past experiences as student and teacher, current practice, and future expected practice. This included: questions/scenarios on ability; nature of tasks, PS, and learning; classroom processes; contexts; planning; and intentions for PS in their teaching. It also included her commenting on five different types of relevant problems without solving them and solving three others while thinking aloud, e.g., "*Two telephone poles of different lengths are to be placed 40m apart. The poles must be anchored by guy wires to a peg located between them. Where should the peg be placed so that the sum of the lengths of the guy wires is minimum?*" Interviews were audiotaped and transcribed. Classroom observations and field notes focused on the teacher's actual instructional behaviours during lessons involving PS. Ten lessons (60 to 85 minutes each) involving PS were observed and audio-taped. Post-observation discussions focused on clarifying the teacher's thinking and actions.

Data analysis involved the researcher and a trained research assistant working indepen -dently to thoroughly review and code the data and identify themes, which were validated through an iterative process of identification and constant comparison. The coding was guided by the 5 components of MPSKT (Table 1). Statements that clearly indicated what Cintia knew for each were highlighted and summarized into subthemes. A separate category was used for relevant coded data that did not clearly fit any of the 5 components and further analysed for emerging themes or components. Data collection was done prior to development of Table 1, independent of this study. So the data were not intentionally based on the 5 components or relationships among them. Thus what emerged from the data reliably represented Cintia's knowledge. Central to understanding how she held this knowledge was examining the data for connections/ relationships in the way she talked about something without being prompted to do so. The connections coded were mapped through charts/diagrams and significance/ strength of them determined by the emphasis, frequency and consistency in occurrence throughout the data. For example, most of her talk about any of the components of MPSKT showed explicit connections to students and her classroom actions were consistent with this. Thus *knowledge of student* emerged as central to how she held and used her knowledge. Further analysis focused on unpacking and confirming the relationships with *students*. Findings reported next focus only on these relationships.

FINDINGS

Cintia held theoretical and practical knowledge of each component of MPSKT (Table 1), but most of it was held as a complex network of relationships among components. Figure 1 is a version of a more complex representation to highlight focus on *students*.

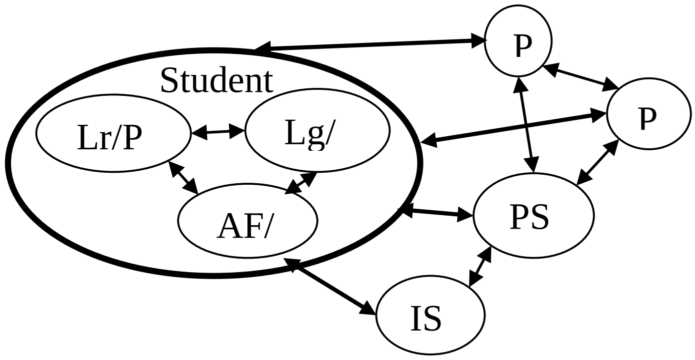

Figure 1: Cintia's connections of components of MPSKT

In Figure 1, Lr/P-Sr is learner/problem solver; Lg/Kg is learning/knowing; AF/B is affective factors/beliefs; P is problem; Pp is problem posing; PS is problem solving; and IS is instructional strategies. The lengths of the connecting lines have no meaning, but the heavier lines represent stronger connections. Descriptions of all of these relationships are obviously not possible here, so only key ones are highlighted next.

Cintia held knowledge of ***students*** in relation to them as learners and problem solvers, their ways of learning/knowing, and affective factors/beliefs impacting them as problem solvers/learners. This knowledge included students as designers, interpreters, evaluators, inquirers and agents of their learning and doing of PS; students' fears, attitude, and beliefs; and expectations for students' learning as in this example.

> Learning occurs when students are able to make meaning about the problems for the concept that has been presented to them, knows why a certain process works and … understand why other ways would not work, … and knows it sufficiently to teach to someone else, talk about it to someone else in their own words.

Of significant importance was the role her knowledge of students played in defining her MPSKT. It formed the core knowledge on which all of the other components of MPSKT in Table 1 depended. It provided an anchor for them and determined how they were enacted in practice. Direct relationship with each is highlighted next.

While Cintia held appropriate knowledge of tasks to support genuine PS, her knowledge of ***problem*** was held as a relationship between the task and the student. For her, a problem emerges based on how the student experiences the task. The task provides possibilities from which this problem emerges, while the student provides the interpretation that gives meaning to the problem, e.g., as interesting, relatable, and challenging or not. Consistent with this, Cintia held her knowledge of ***problem posing*** in relation to students as task designers. Thus, problem posing involved students creating tasks, modifying/redesigning tasks contextually, and extending a problem. It allowed them to realize their interests and be creative. As she explained: "A lot of times, the problems are generated from the class." "Students have opportunities to write problems on their own about things that they find interesting to share with others." She also required that they produced solutions and grading schemes for their problems as a way to understand PS.

Cintia is proficient in PS and has knowledge of general heuristics/models of PS (e.g., Polya's (1957) model), but she held her teaching knowledge of ***problem solving*** in relation to her knowledge of students. For her, PS is determined by the students' thinking/inquiry process. It is what students are able to do that makes sense to arrive at a solution. She explained, "I make sure I tell them that I don't care how they solve it. It doesn't make any difference what kind of methodology they use as long as it's logical and they can explain it and let other people understand it." She generalized this as:

> It's like anything else that you don't know what the outcome will be and you're kind of game for anything else, so you just take your chances and you try and use the tools that are available to you, see what happens. … It is whatever takes you to get to that solution. So thinking, trying out things, writing, using whatever tools are necessary to find a solution.

Finally, strong bond between knowledge of ***instructional strategies*** and knowledge of student made Cintia's teaching of PS about empowering students and not the teacher. She held knowledge of instruction to support students' agency and autonomy in PS. This included not teaching general or specific heuristics in an explicit way, but allowing them to emerge out of students' experiences in trying to solve a problem and reflecting on the process. It also included the importance of having "students work with others so that ideas could bounce back and forth between them and … have that opportunity to do problems individually" and "students understand how they're going to be graded." She also held knowledge of productive struggle and how to facilitate it, knowledge of when and how to intervene, and knowledge of questioning. For example,

> I go around and listen to the groups. …I can sit next to any group and they talk, and I ask them questions if they're stuck but that's about it. I simply watch how the groups are working together and if I see a group is stuck, I try to come up with a question that will allow them to continue, but I will not give anybody the answer at any time …they can always ask a question, but if they want to know how to do it, or are they right, they may not talk to me.

> I have the students usually in three's working on a different problem then … they present the problem to the class, and because they're all different, the kids have something interesting to listen to and to learn from.

Cintia's knowledge of instruction is also held in relation to ***affective factors*** associated with the students. Her knowledge included: how to help them "to not be afraid to try different kinds of things;" the importance of allowing them to "work in groups as well as individually so that they could see lots of problems and be comfortable doing them;" when and how to allow them to have a choice of problems to solve on assignments and tests because "giving them a choice often helps a lot of times;" and the importance of

> getting them used to reading and writing mathematics problems on their own so that they're desensitized, if you will, to word problems and begin to see them as experiences where they can struggle and the struggle is part of the process, then it's not a scary thing but the struggle is meant to help them arrive at an answer.

CONCLUSIONS

The study indicated that MPSKT from the teacher's perspective includes a complex network of interdependent knowledge. Thus, while the categories of Table 1 provide a way of making sense of MPSKT in a general way, the study highlights the possible importance of how they are held by teachers to be usable in a meaningful and effective way in supporting PS proficiency in their teaching. It suggests that there is a core or anchoring category of knowledge that gives meaning to the other categories of knowledge in MPSKT and is a critical factor in how PS is taught. In Cintia's case, having knowledge of students as the core knowledge enabled her to empower students in learning PS. Thus from a practice-based perspective, there could be different core knowledge that teachers hold with different impact to teaching PS. Future studies are needed to understand other types of core knowledge held and how they work or not work to support students' engagement in PS in order to help teachers to understand how they hold MPSKT and develop appropriate core knowledge for teaching PS.

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SNAPSHOTS OF A TEACHER'S IN-THE-MOMENT NOTICING DURING A LESSON ON GRADIENT

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This paper introduces the notion of snapshots of noticing, which captures the noticing processes of teachers to characterise their mathematical noticing. I present two snapshots of a teacher's in-the-moment noticing when she interacted with a Secondary One student during a lesson on gradient in Singapore and illustrate how snapshots of noticing can provide an analysis of a teacher's noticing during the lesson.

ANALYSING TEACHER NOTICING

There has been a growing interest to decompose and analyse teaching to make teaching practice more learnable (Ball, Sleep, Boerst, & Bass, 2009; Grossman & McDonald, 2008). The idea is to move away from thinking of teaching as entirely improvisational to a more balanced view, where important skills and routines can be mastered even by novice teachers (Ball & Forzani, 2009; Lampert & Graziani, 2009). Mathematics teacher noticing, which refers to what teachers see and how they interpret their observations to make instructional decisions, is one of these core skills (Jacobs, Lamb, & Philipp, 2010; Mason, 2011). Teacher noticing is seen as an important component of teaching expertise, and it has the potential to improve teaching practices (Mason, 2011; Schoenfeld, 2011). Moreover, to bring about a pedagogy that focuses on developing student reasoning, Erickson (2011, p. 33) contends that it is important for mathematics education researchers to "learn more about the what, how and why of teacher noticing". Hence, it is timely and useful to analyse what and how teachers notice.

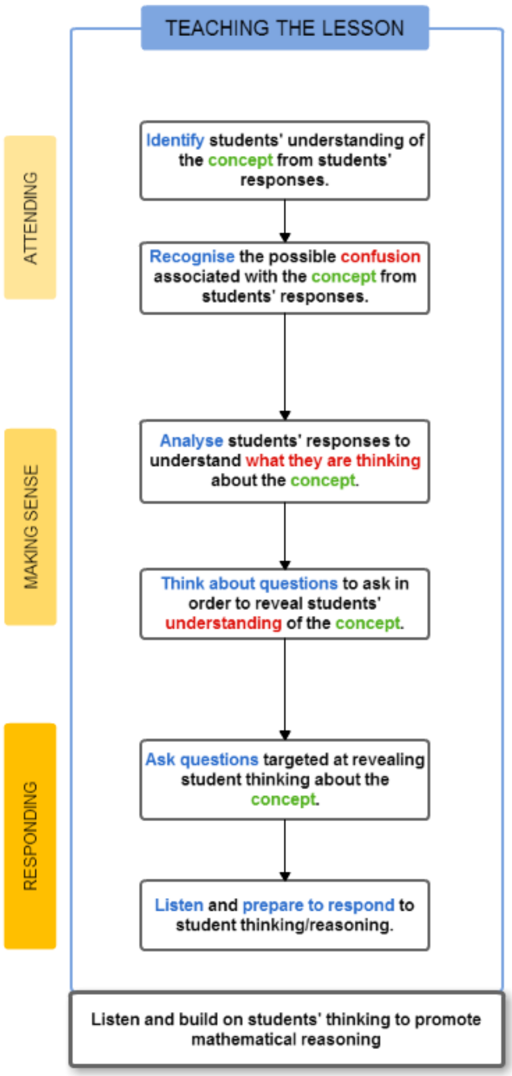
However, analysing what teachers notice during a lesson poses some challenges. First, despite the apparent simplicity of the construct of teacher noticing, the ability to "notice productively" during mathematics teaching is both difficult to master, and complex to study (Jacobs, Philipp, & Sherin, 2011, p. xxvii). Next, as Mason (2011) argues, it is difficult to track what teachers attend to given that one can attend to different things at various levels of details simultaneously. Although Sherin, Russ, and Colestock (2011) had used wearable cameras to access and study teachers' in-the-moment noticing with some success, they also acknowledged that teachers' attention and thinking could only be better accounted for during the interviews after lessons. Moreover, to support teacher in developing their noticing expertise, it is also crucial to characterise teacher noticing in a way that honours the complexity of classroom practice, while at the same time, pinpoints specific actions that teachers can take. In light of the above discussion, this paper introduces one such framework to characterise teacher noticing that is productive for enhancing students' reasoning, illustrates its application in examining

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 147–154. Szeged, Hungary: PME.

how a teacher's noticing can be described, analysed and represented using snapshots of noticing, and discusses the possible implications of this framework.

THE FOCUS FRAMEWORK: CHARACTERISING PRODUCTIVE NOTICING

Using a photography metaphor to describe and analyse the complexity of teaching is not new. For example, Lampert (2001) contends that a teacher has to actively zoom in and zoom out, across time and relationships, to focus on different aspects of teaching. The FOCUS Framework, developed as part of a doctoral study (Choy, 2015), highlights two critical dimensions that promote productive noticing: The need for an explicit focus for noticing (what to notice) and the central role of pedagogical reasoning (how to notice).



Drawing on the Three Point Framework described by Yang and Ricks (2012), the FOCUS Framework highlights three mathematically significant aspects for teachers to direct their noticing—the concept, confusion, and course of action. Furthermore, it is also crucial for teachers to align their teaching approaches (courses of action) to target students' learning difficulties (confusion) associated with the mathematical concepts. However, this alignment is not automatic and is mediated by the teacher's pedagogical reasoning. Therefore, when teachers analyse their observations and provide the evidence or justification for making an instructional response, they are more likely to generate an instructional decision that promotes students' reasoning.

A theoretical model of noticing has been developed from the FOCUS Framework to describe what, and how, a teacher can notice productively when learning from practice (See Choy, 2015, p178.). It maps a teacher's noticing processes (attending, making sense, and responding) through three stages of learning from practice (planning, teaching, and reviewing) to the three key productive practices for mathematical reasoning (designing lesson to reveal thinking; listening and responding to student thinking; and analysing student thinking). In other words, the model describes a theoretical process of productive noticing, which highlights explicitly the three crucial focal points, and how the alignment between these three points can be achieved. An example of how the model describes productive noticing during teaching is shown in the figure above. By comparing a teacher's noticing to the theoretical model, a snapshot of noticing can be developed to provide a picture of what, and how, the teacher notices during a teaching segment.

SNAPSHOTS OF NOTICING: DESCRIBING ANITA'S NOTICING

A snapshot of noticing depicts what a teacher sees, and how, one interprets the observations to make instructional decisions. This new notion is different from the levels of noticing expertise developed by van Es (2011), and does not give a static score to assess teachers' ability to notice (Jacobs et al., 2010). Instead, it offers a new and dynamic perspective of teacher noticing by capturing the flow of noticing processes from lesson planning to lesson review after teaching. In this section, I will demonstrate how the model can be used to develop snapshots of a teacher's noticing during her teaching.

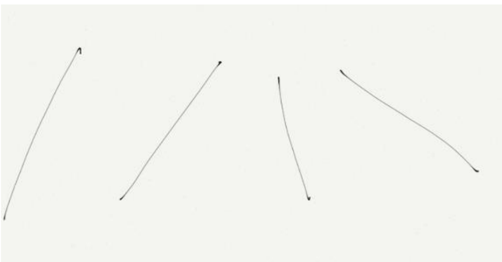
Context of Anita's lesson on gradient

Anita, a mathematics teacher with 12 years of teaching experience, was part of the study from which the FOCUS Framework was developed. In this vignette, she was teaching a lesson on gradient for Secondary One (aged 13) students in a Singapore school. In the context of this study, gradient is defined as the ratio of the vertical change to the horizontal change. Students, who encountered the coordinate system for the first time, were expected to find gradients of straight lines set in the Cartesian coordinates system (1 cm represents 1 unit) without the formula for computing the gradient of a straight line using coordinates.

Episode 1: Missed opportunity to orchestrate a discussion

In this episode, Anita realised that students mistook height as gradient during the lesson (See Lines 30 to 33 below). Although she started off with a potentially illustrative example, Anita did not take the opportunity to initiate a discussion, and instead decided to tell students that slope is rise over the run (See Line 36):

30. Anita: (Draws four lines) But lines can be this way… something like that or something like that.



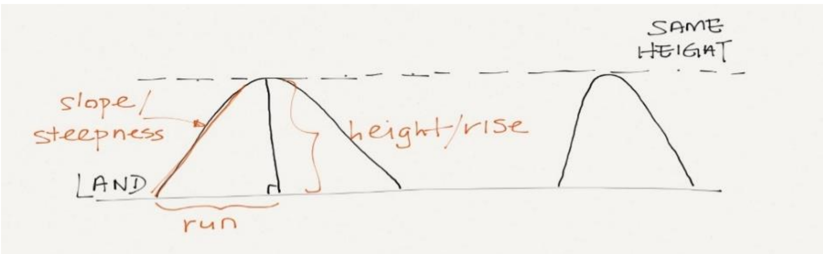
So we notice… they have different? Steepness. Another word for steepness is? [Without waiting.] Slope… And talking about the steepness… how is it applicable in our daily lives? How can you see the link? If you notice, when you climb up a mountain, or go up the staircase… Here's a mountain… (Draws a mountain.) If you are climbing up the mountain, the bigger the number, what do you notice about the slope?

31. Students: Higher.

32. Anita: Higher? You mean higher mountain when the number is bigger?

33. Students: (Various answers. Cannot be transcribed.)

34. Anita: So, if you mean that if a mountain that is higher, and this mountain is higher, the gradient is even higher? Remember? The key word here is the steepness. So, let's say I have two mountains… The two mountains are of the same height. One mountain is like that… the other mountain … I'm drawing here. This is the base… the land. They have the same land [*sic*] right? And this is the top. When we are learning gradient, right? What do you think we are learning? So, we are learning about how high is the mountain?



35. Students: How steep?

36. Anita: How steep is the mountain? It's not a matter of the height. It's about how steep. So, we are talking more about the slope, or what we call steepness. In order to find the gradient, which is the steepness…in order to find how steep, we are going to use? The? Height over the? Horizontal. So, we are looking at the steepness here, right? I need to use the reference from the height, which in this case is the? Rise. Because we are going to address how steep is the slope right, we will look at how high it is…

In this excerpt, Anita attended to her students' confusion—a slope that is steeper is higher (Line 34)—from their responses (Line 31). Anita demonstrated her awareness of the students' confusion by drawing two mountains, of the same height but with different gradients, to illustrate the concept of steepness (Line 34). Her response suggests that she might have analysed her students' confusion in-the-moment, because this scenario was not discussed during the planning sessions. While the analogy of mountains may not correspond directly to straight line graphs, Anita possibly assessed that the use of these two figures would draw students' attention to the relationship between steepness, gradient, height (rise) and the run.

However, Anita was more focused on getting students to know the important terms and calculations than orchestrating a discussion to listen to their thinking. She told the students that calculating gradient is the same as finding "the height over the horizontal" (Line 36). Her explanation only revolved around computing "height" or "rise", and she left out the relationship between the "rise" and "run". Even when Anita later explained the difference between a steep slope and a gentle slope, she assumed that students could see the *left-to-right* convention, and that a steeper slope corresponded to a "bigger number". Furthermore, her questioning was more evaluative, requiring students to give a closed answer ("How steep." see Line 35). Therefore, evidence suggests that while Anita noticed student thinking, she chose to use a more teacher-directed mode of instruction (Lines 34 and 36), which was not productive with regard to promoting students' reasoning. Although Anita was able to bring to her mind a counterexample to illustrate the point about steepness and height, she opted for telling, which did not provide opportunities for students to reveal their understanding. Thus, her response was not targeted at her students' confusion and hence, her noticing, as a whole, was not productive.

Episode 2: Telling instead of listening

Anita's preference for telling students during teaching was also clearly seen when she interacted one-to-one with the students. For example, when Student S4 asked Anita about the determination of "rise", she reacted by telling without listening to find out what Student S4 was thinking about:

38.	S4:	Ms Anita, what's the rise? How to count?
39.	Anita:	The rise is the height. You find the perpendicular, and count how many units are there here.
40.	S4:	So, you count by the?
41.	Anita:	Yes. You count by the boxes. How many boxes are there? (Moves to the front of the classroom to explain how to count the "rise".)

Anita's definition (Line 39) requires Student S4 to see the same right-angled triangle as hers. The most natural triangle in the problem of interest is the biggest triangle formed with the end points of the line segment. Anita did not ask what Student S4 was specifically puzzled about with regard to the "rise". Instead, Anita's emphasis on "counting" might cause Student S4 to associate the "counting of boxes" with the measurement of "rise" and "run". It was not clear whether Student S4 understood the key idea, that what matters is the ratio of the "vertical change" to the "horizontal change" at this point. However, it became apparent that Student S4 had difficulties with Question 1b (See Figure 1) when coordinates were introduced a few minutes later.

She was confused about how the "rise" and "run" are counted when the coordinates involved were negative.

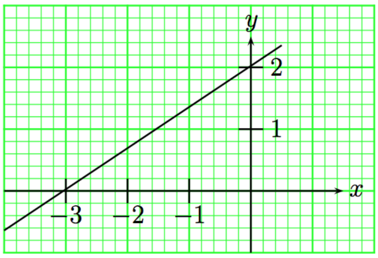


Figure 1: Question 1(b) in the worksheet.

Like several other students, Student S4 mistook the run as "negative three" because they took reference from the origin. Anita asked Student S4 to ignore the origin and directed her to focus on "forming the right-angle triangle" and "counting the rise and run". However, she did not make explicit how distance is measured or calculated in a coordinate system, and did not realise that her students might have problems when coordinate were introduced.

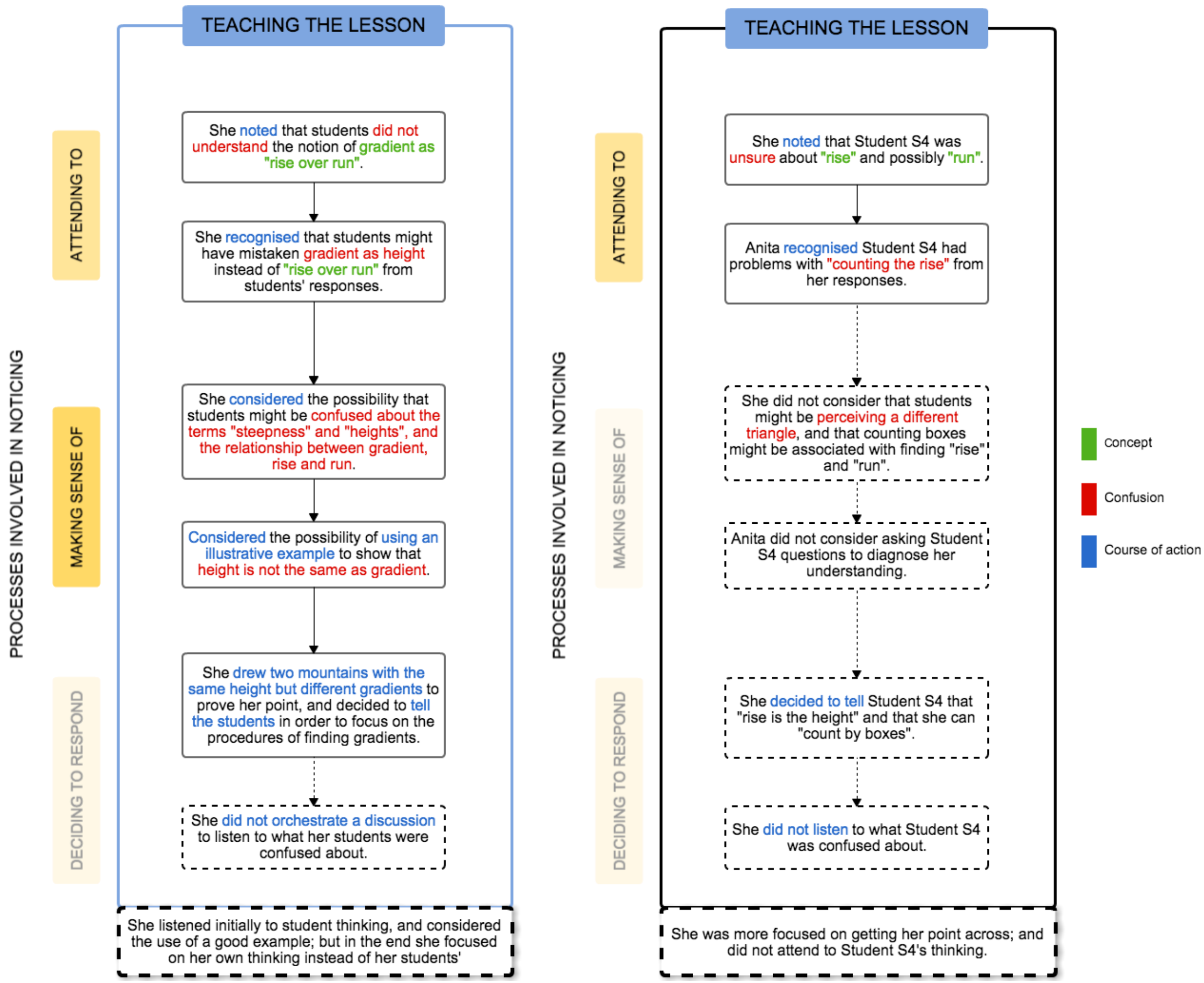


Figure 2: Snapshots of Anita's noticing during teaching.

Snapshots of Anita's noticing: What do they tell us?

Anita's noticing during these two episodes may be summarised as follow:

1. Anita attended to an unexpected difficulty of confusing gradient with height (Episode 1); and the difficulty of finding the rise (Episode 2);
2. Even though she might have analysed the difficulty as evidenced by the use of the mountain example, Anita still decided to tell the students (Episode 1); and
3. Without preparing to listen to her students' reasoning, Anita's instructional strategy remained consistent (telling) through the lesson, which might not have targeted the difficulty that surfaced (Episode 2).

Snapshots of her noticing processes during the two episodes are represented in Figure 2 (The left for episode 1 and the right for episode 2). These snapshots present a clear picture of what, and how, Anita noticed during the two teaching episodes. As seen from the snapshots, while Anita had attended to the specific details with regard to her students' confusion (both episodes), she might not have interpreted her students' thinking (episode 2), and missed opportunities to build on students' reasoning (both episodes). These observations suggest that Anita did not align her responses to target students' confusion, and hence her noticing is characterised as non-productive according to the FOCUS Framework. More importantly, the snapshots highlight that Anita seemed to focus more on her own thinking, and as a result, did not ask her students questions that might have revealed their thinking. By comparing Anita's snapshots with the theoretical model, it is therefore possible to pinpoint specific actions that she could take to raise her noticing expertise, and potentially improve her classroom practices. Last but not least, a portrait of Anita's noticing can be formed by putting together her snapshots of noticing. This can provide researchers a means to observe the regularities in her noticing and account for these patterns in light of the analysis of her teaching episodes.

CONCLUDING REMARKS

This paper offers snapshots of noticing as representations of practice (Grossman & McDonald, 2008), which can be used to discuss and analyse teaching and its interactions with the processes of noticing. As a theoretical model, the FOCUS Framework characterises the notion of productive noticing and enables investigations of teaching while preserving its complexities. Besides its value as an analytical tool, the theoretical model can also serve as a self-reflection tool for teachers by directing their attention to mathematically significant aspects of their teaching. How this can be realised in teacher education and professional development will be a fruitful area of research in the study of teacher noticing. Furthermore, the model described in this paper was developed in the context of promoting student reasoning. It remains to be seen whether this model can be applied and tested in other contexts, such as teaching mathematics in a technologically-enhanced environment.

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EXAMINING MATHEMATICS TEACHERS' JUSTIFICATION AND ASSESSMENT OF STUDENTS' JUSTIFICATIONS

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This paper presents an analysis of teachers' justification to a geometry task concerning the converse of Pythagoras' theorem and their assessment of two authentic student justifications to the same task. Data were collected through administering a written test and a questionnaire combined to 50 mathematics teachers from over 30 secondary schools. The findings revealed that teachers were able to produce the correct working although many did not cite the right reason or any reason at all. When assessing students' justifications, the teachers did not seem to be clear about the rigour of justification and tended to be lenient in scoring if the reason was not explicitly stated.

BACKGROUND

In recent years, educators and school reformers worldwide have called for schools to develop in students a broad set of competencies such as character traits, knowledge and skills that are believed to be an imperative for success in the workplaces in the 21st century. Given the emphasis on 21st century competencies (21cc), greater demands are then being placed on students to reason, explain and justify in the learning of mathematics. In Singapore, mathematical reasoning and justification should have been carried out in most mathematics lessons because, for many years, *reasoning* and *communication* have been two key process skills in the Singapore Mathematics framework (Ministry of Education (Singapore), 2012). But the problem is the extent of them being practised is not clear! The reports for the national examinations in Singapore have frequently revealed that justifying mathematical claims not in the context of formal proof is fraught with difficulties. To find out why students fail to establish justifications correctly, a small exploratory study was first conducted on Singapore mathematics teachers to examine their ability to justify non-proof tasks. This paper reports on the performance of the teachers in this study.

JUSTIFICATION IN MATHEMATICS

The topic of justification is often associated with the topic of proof in the literature. According to Simon and Blume (1996), mathematical justification is the process of "establishing validity [and] developing an argument that builds from the community's taken-as-shared knowledge" (p. 28). This notion of justification as a means of determining and explaining the truth of a mathematical conjecture or assertion resonates strongly with many other researchers such as Balacheff (1988), Thomas (1997), Harel and Sowder (2007), and Huang (2005).

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 155–162. Szeged, Hungary: PME.

The types of responses expected of students in the justification process depend on at least two factors: *the cognitive abilities of students* and *the nature of the task*. For secondary school students, particularly those in the lower grades, a justification does not need to measure up to a formal proof. This is because providing a theoretical argument for a mathematical result is sometimes not required in the light of their cognitive level until they reach higher level of study (Hoyles & Healy, 1999). Take, for instance, the justification task asking lower secondary school students why $2n - 1$ is an odd number for any positive integer n. An acceptable justification could simply state: with n being a positive integer, putting together two groups of n, which is $2n$ when expressed in notation, thus forms an even number, therefore subtracting one from it will result in an odd number.

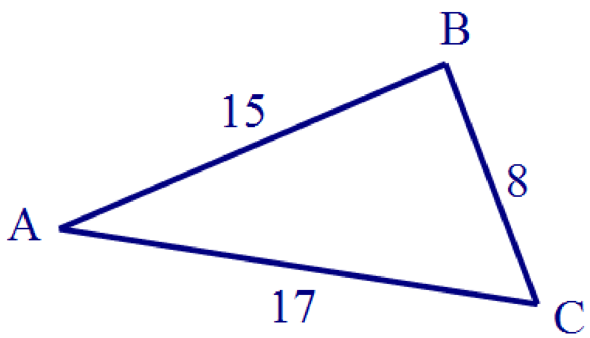
Certain justification tasks lend themselves well to experiential justification, which is mainly supported by specific examples and illustrations. Take, for instance, the task asking why the rule $a^m \times a^n = a^{m+n}$ is true for any positive integer a, m and n. Students can rely on intuitive reasoning through using several numerical examples in the justification. Such a justification does not involve any established theorems and is therefore deemed a less formal argument than a typical mathematical deductive proof (Becker & Rivera, 2009). But it is this type of justification that is valued by educators because it "explains rather than simply convinces" (Lannin, 2005, p. 235).

Justification tasks such as the two examples provided above are quite different in nature from the typical proof questions in that they do not require the use of theorems to establish the validity of the mathematical claims. After reviewing the research literature and Singapore past years national examination questions, it was found that the justification tasks can be classified into four broad categories: *validate* (e.g., explain why $2n - 1$ is odd), *elaborate* (e.g., explain how you obtained $2n - 1$ as a general term of a number sequence), *interpret* (e.g., explain what the y-intercept of the graph represents), and *predict* (e.g., explain which of the three averages – mean, mode or median – is the most suitable for the data set). The present study used justification tasks such as these to examine the mathematics teachers' ability to justify and their assessment of students' justifications. The two research questions that are addressed here are: How do Singapore mathematics teachers justify a geometry task concerning the converse of Pythagoras' theorem? How do the mathematics teachers assess students' justifications?

METHODS

A 30-minute written test-survey consisting of two parts was administered to 50 mathematics teachers from 32 different secondary schools in Singapore. 30% of the teachers have taught mathematics for fewer than 5 years, another 40% for at least five to less than 15 years, and the remaining 30% for over 15 years. Part 1 of the instrument comprised four justification tasks, one each in number, algebra, geometry and statistics. Part 2 comprised the same four justification tasks, this time each accompanied by two authentic student solutions. The teachers had to individually complete all four justification tasks and the marking of eight student solutions. For Part 1, the teachers had to construct the justification that would deserve the best mark. For Part 2, since a justification task typically carries one mark in the national examinations, the teachers were asked to score the authentic student solutions using a dichotomous scoring scale with 1 point for a correct response and zero for an incorrect response. Only the geometry task in Figure 1, called *Mr. Right Triangle*, concerning the converse of Pythagoras' theorem is reported here.

In triangle ABC, AB = 15 cm, BC = 8 cm and AC = 17 cm.



Explain why angle ABC is a right angle.

Figure 1. Mr. Right Triangle

Figure 2 below presents the two student solutions for *Mr. Right Triangle* given in Part 2. Produced by Year 9 students in another project, these solutions are believed to represent common and unacceptable justifications. In Student solution 1, the values of AC^2 and $AB^2 + BC^2$ were determined separately and found to be equal. But Pythagoras' theorem should not be cited as a warrant when drawing the conclusion that "it is an right-angle triangle". In Student solution 2, the warrant for the justification: *the formula* $a^2 + b^2 = c^2$ *can only be used for right-angled triangle*, is actually a rephrasing of the Pythagoras' theorem that the student was very clear about (see his first statement). So like Student solution 1, this justification was also unacceptable on grounds of incorrect warrant.

Student solution 1	Student solution 2
$AC^2 = AB^2 + BC^2$ $17^2 = 15^2 + 8^2$ $289 = 289$ when we use phythogar's therom when AC^2 is equal to 289 $AB^2 + BC^2$ is also equal to 289 therefore it is an right-angle triangle.	Pythagoras theroam right angle △ = $a^2+b^2=c^2$ $a = 15$ $b = 8$ $c = 17$ $\sqrt{15^2+8^2} = 17$ the formula $a^2+b^2=c^2$ can be used for and only ⊾△ Since $a^2+b^2=c^2$, △ABC is a right angle △ #

Figure 2. Student solutions for *Mr. Right Triangle*

All the teachers' test scripts were collected and immediately coded T1 to T50. The teachers' responses to *Mr. Right Triangle* in Part 1 were analysed carefully. A correct justification must ascertain that the condition $8^2 + 15^2 = 17^2$ is satisfied followed by citing the *converse* of Pythagoras' theorem as a warrant for drawing the conclusion that angle ABC is a right angle. The teachers' justifications were analysed again two days later by the researcher and 100% consistency was achieved. As for the teachers' assessment of students' justifications in Part 2, a frequency count was done for each given student solution to determine the number of teachers who awarded it one point and zero.

RESULTS AND DISCUSSION

This section addresses the two research questions by reporting what has been found from the analyses of data.

(1) How do Singapore mathematics teachers justify a geometry task concerning the converse of Pythagoras' theorem?

Five categories of responses as presented in Table 1 were observed amongst the mathematics teachers' justifications.

Code	Description
CP 5	Correct working with correct warrant
CP 4	Correct working with incorrect warrant
CP 3	Correct working with no warrant
CP 2	Partially correct working
CP 1	Wrong working

Table 1: Types of teachers' justification for *Mr. Right Triangle*

Mr. Right Triangle appears somewhat challenging for the mathematics teachers. Only 14 of them were successful in giving a clear and complete justification (CP5). On closer examination of the unsuccessful teachers' responses, it was discovered that of the 36 incorrect responses, 22 of them were coded CP4 (e.g., see Figure 3a), four were coded CP3 (e.g., see Figure 3b), and another seven were coded CP2. Only three mathematics teachers produced a completely wrong justification.

In Figure 3a, T50 established the condition $AB^2 + BC^2 = AC^2$ by separately working out the values of AC^2 and $BC^2 + AB^2$, and noticing that both values were equal. Subsequently, the teacher inferred that triangle ABC is a right-angled triangle and the angle opposite the longest side AC (i.e., ∠ABC) is 90°. This justification seems logical and systematic except that it is actually flawed. The correct warrant to use should be the *converse* of Pythagoras' theorem and *not* Pythagoras' theorem. On the other hand, the justification by T47 in Figure 3b is less precise than the one produced by T50. This teacher never stated any warrant to substantiate his justification.

These results indicate that a significant number of teachers did understand what was required to deduce their conclusions: that is, to check that the condition $8^2 + 15^2 = 17^2$ holds before substantiating the claim with the *converse* of Pythagoras' theorem. But the compelling evidence of nearly half of the teachers citing the wrong warrant in their responses points to a possible misconception amongst them: that the Pythagoras' theorem and its converse might have been regarded as essentially the same – an observation similarly noted by Wong (2015) as well.

(a) Solution by Teacher T50

Explain why angle ABC is a right angle.

$AC^2 = 17^2 = 289$ $\quad BC^2 = 8^2 = 64$ $\quad AB^2 = 15^2 = 225$

$AC^2 = 289$

$BC^2 + AB^2 = 64 + 225 = 289$

$\therefore AC^2 = BC^2 + AB^2$

By pythagoras' theorem, ΔABC is a right ∠ Δ. Since AC is the longest length;

∴ ∠ABC must be ~~the~~ 90°.

(b) Solution by Teacher T47

$8^2 + 15^2 = 64 + 225$
$= 289$
$= 17^2$

Hence ∠ABC is a right angle.

Figure 3. Teachers' justifications for *Mr. Right Triangle*

(2) How do the mathematics teachers assess students' justifications?

Table 2 presents the distribution of the mathematics teachers' scores for each of the two authentic student solutions by the types of justifications they produced. It shows that there are variations in the distribution of teachers' scores between the two student solutions. Sixty percent of the teachers awarded zero to Student solution 1 but the corresponding percentage of teachers for Student solution 2 dropped to 40%. The analysis appears to suggest that a considerable number of teachers were able to recognise that Student solution 1 was flawed because Pythagoras' theorem was mentioned and they knew it was the wrong warrant. When Pythagoras' theorem was not explicitly stated as a warrant in a similar solution such as Student solution 2, the teachers were divided in their assessment, with more teachers awarding one mark. This finding demonstrates on one hand that the teachers tend to be more lenient and likely to accept the response as correct when it is not, but on the other hand, it appears that the teachers are not clear about the rigour of justification of this nature.

(n = 50)	Student solution 1		Student solution 2	
score code	0	1	0	1
CP 5	11	3	8	6
CP 4	14	8	9	13
CP 3	1	3	-	4
CP 2	4	3	2	5
CP 1	1	2	2	1
	31	19	21	29

Table 2: Teachers' scores by the types of teachers' justification

When the teachers' scores for the two student solutions are compared with their justification, two features of particular interest are noted. One is the assessment by the 14 teachers who produced code CP5 responses. These teachers ought to know very well the correct justification for *Mr. Right Triangle* and should be able to spot the mistakes in the two student solutions. Yet nearly half of them accepted Student solution 2 as correct. A second striking feature is the higher frequency of zero mark than one mark in Student solution 1 by teachers who constructed code CP4 responses. Student solution 1 was similar to what they produced, yet it was rejected. The reasons for this are not yet known though interviewing teachers might offer some clues.

CONCLUSION

Mathematics as a discipline calls for an examination and evaluation of the validity of facts, articulation of reasons for employing a certain method to solve a mathematical task, and substantiation of any arguments put forth. So justification is a crucial process skill enabling all these activities to be carried out. But the justification process appears to be complex. Even some mathematics teachers fail to navigate this process successfully. The present study, though small in size, is certainly worthwhile to develop further. Despite the limitations, it is hoped that the findings presented here offer useful ideas for researchers to think about mathematics teachers' understanding of justification and ability to justify.

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FACILITATING DISCUSSION OF VIDEO WITH TEACHERS OF MATHEMATICS: THE PARADOX OF JUDGMENT

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This report details findings, related to the role of the facilitator, from a UK government-funded project to promote the use of video clubs for the professional development of teachers of mathematics. Seven teachers met on six occasions, over a three-month period, and shared video recordings of their own classrooms, all meetings were themselves recorded. While it is a common finding that discussion norms can be hard to establish, participants adapted to intended norms from the first meeting. The way this was achieved is analysed, within the enactivist methodology of the project. There is an apparent paradox that a move away from judgment is achieved through the use of judgment. Bateson's (1972) levels of learning and communication are offered as one explanation of the observed phenomena.

INTRODUCTION

This report details results, relevant to the role of the facilitator, from research into the professional learning of teachers of mathematics, funded by the UK's Economic and Social Research Council (ESRC). The aim of the research is to investigate effective use of video for teacher learning, building on previous work conducted by the Principal Investigator (Coles, 2013, 2014) and to promote and support such use via the creation of 'video clubs' in the UK and beyond.

The video clubs, as conceived in this research, last over a three-month period with participants meeting fortnightly on six occasions. The clubs are partly inspired by those run in the USA (e.g., van Es et al., 2014). This report is based on outcomes from a video club that ran between May and July 2015. There were seven participants (all volunteers). Participation in the club is free, but a commitment is required to attend and engage in activities (most importantly video recording of their own classroom) between meetings. The number of participants could be up to ten, based on principles of collaborative group-working (Brown and Coles, 2011). The video club was framed around an action research text (Altrichter, Posch and Somekh, 1993) and participants were asked to come to the first meeting having read the first chapter and engaged in an activity (from the book) to help them find or refine an issue in their teaching they wanted to develop or investigate. In other words, the video clubs were not set up with any particular pedagogical focus in mind, but instead with the aim of supporting each participating teacher in developing their own practice.

After a review of literature on the use of video, the enactivist methodology of the project is described briefly. Results are then presented and analysed, prompting further theoretical discussion in order to offer an explanation of those results.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 163–170. Szeged, Hungary: PME.

USING VIDEO

In reviewing the history of the use of video for teacher learning, Sherin (2007) cites reports going back to 1966 that found mixed results in terms of effectiveness (McIntyre, Byrd and Foxx, 1966). Sherin (2007) suggested more empirical work was needed to gain appreciation of possible roles for video – work that she herself has continued to conduct (e.g., Sherin and van Es, 2009). It is only relatively recently, however, that there has been a sustained interest in investigating the role of the facilitator of discussion, when working on video with teachers (Borko, et al., 2011; Coles, 2013; van Es, et al., 2014; Zhang et al., 2011). This report contributes to the emerging field exploring the skills needed to facilitate discussion.

There are many methods proposed for using video with teachers (e.g., Star and Strickland, 2008; Santagata and Angelici, 2010). My own use of video is drawn from Jaworski (1990) and involves using 3-4 minute clips, where the first task for participants is to reconstruct (without interpretation or judgment) what happened, before any move to accounting for events is allowed (see Coles, 2013).

More recently, frameworks have been suggested to support facilitators. Due to space limitations, just two of these frameworks will be described briefly below (chosen for their relevance to this study). At present, a common (and unsurprising) feature of frameworks relevant to facilitators is that they come out of particular uses of video. This report attempts to draw out issues around the facilitation of video that are potentially independent of the particular method of video use being employed.

van Es et al., (2014) analysed discussion in their own video clubs and identified four categories that represented key strategies used by experienced facilitators during high quality discussions (defined as those discussions where there was sustained engagement in making sense of students' thinking or participants' own thinking). These four categories are: orienting the group to the video analysis task; sustaining an inquiry stance; maintaining a focus on the video and the mathematics; supporting group collaboration (van Es et al., 2014, p.347). A list of decision-points for the facilitator in Coles (2013) shares many aspects of the van Es et al., framework. In Coles (2013), the following five decision-points are identified: selecting a video clip; setting up discussion norms; re-watching the video clip; moving to interpretation; metacommenting.

Common to both frameworks is: (i) the importance of establishing a mode of talk, whatever that mode is, (orienting the group to the task; setting up discussion norms); and, (ii) helping participants make links to their own practice (sustaining an inquiry stance; moving to interpretation). One other issue relevant to this report that has arisen from a recent review of video use, not just within mathematics education (Gaudin and Chalies, 2015), is the cognitive load of video viewing and the problems, particularly with beginning teachers, in their 'capacity to identify and interpret classroom events' (Gaudin and Chalies, 2015, p.29).

ENACTIVISM AS A METHODOLOGY

The methodology behind the project was enactivist (Reid and Mgombelo, 2015). Enactivism is a research stance that offers a way through the philosophical and practical pitfalls of the subject-object divide, and all it entails, through collapsing knowing, doing and being (Maturana and Varela, 1987). From an enactivist stance, 'all doing is knowing and all knowing is doing' (ibid, p.27). What we think of as 'subject' and 'object' arise together in patterns of co-ordinated activity, each one (co-)determining the other. All perception is an active process – perhaps easy to acknowledge with touch but maybe harder to notice with vision.

We say that someone (including ourselves) knows something if we observe them acting in an 'adequate' manner in an environment. Knowing is therefore never fixed, never certain and alters in each expression. Knowing cannot be separated from acting and our whole being. We acknowledge learning when we observe someone acting differently in a similar context (perhaps moving from inadequate to adequate action).

The implications of the enactivist stance for the doing of research were explored in a special issue of *ZDM, The International Journal of Mathematics Education*, volume 47, issue 2. Of relevance here is an enactive approach to studying language (Coles, 2015), since the data from this project is recordings of talk. Analysis was conducted beginning with a search for patterns in the last piece of data collected, on the principle of equifinality (see Coles, 2015 for one description of this principle). In this project, in the final meeting of the club, participants were invited to reflect on what they had learnt and anything significant they would take away from having attended meetings. A pattern observable during this conversation was that *every* participant mentioned something related to questioning their own immediate 'judgment' of situations, or the difficulty of not interpreting events to fit one's ideas. A random selection of comments is below (phrases linked to judging are underlined).

> 'From that very first session when we watched that video and I think that's the one thing I've picked up most from this club is understanding how <u>you doctor what you watch unintentionally</u>' (Teacher N)

> '<u>Just that judgment, being judged and judging</u> … After we watched that first [video] … <u>we were making judgments</u> … but then that wasn't really reflection' (Teacher J)

> 'At the very beginning I found it so difficult just to <u>be objective</u> and I have realised that this is a direct reflection of how I am in the classroom. I listen to children and sometimes <u>I don't listen</u> to the question for the question's sake, and move it on, trying to keep that pace high.' (Teacher T)

Having identified a theme, the enactivist approach in Coles (2015) is to trace mention of this theme through earlier sections of data. The aim is to follow the emergence of the theme rather than account for patterns in a directly causal manner. So, starting at the first meeting any mention of judgment, or difficulties with interpretation, in the audio recordings were transcribed; the aim is to uncover further patterns related to the role of the facilitator (which is the focus of the research project).

FACILITATING THE MOVE AWAY FROM JUDGMENT

A striking feature of the comments reported above (from meeting 6) is the number that refer back to the first meeting. It appeared as though a significant shift took place during the first meeting, in relation to the move away from judgment. This first meeting would have been the starting point for analysis anyway, and in this section the focus is just on that meeting. Three transcripts are reported, which are all the instances where a comment from a participant gets interrupted or re-focused by the facilitator, in relation to judging/interpreting. Given the focus of this report on the decision-making of the facilitator (in this case, me), the three transcripts are presented in the form of a narrative, combining what was said with my own, stimulated-recall of the events. After the three transcripts/incidents I offer some further reflections.

Incident 1

In the first meeting, having had some time discussing how the group would operate and hearing what participants had done on the pre-meeting tasks, we moved to watch our first video. No participants were expected to take video recordings of lessons before this meeting and so I chose a video clip from the Video Mosaic database (www.videomosaic.org) called 'Alan's Infinity'. I have used this clip before and am aware it can provoke strong responses (both positive and negative) and so hoped it would be suitable to establish the discipline (Jaworski, 1990) of starting work on video with the detail of what took place, without initially straying into interpretation.

I was explicit that the initial task would be to simply say what participants saw on the video. I let the video run, pressed stop and as I was returning to my seat one teacher (P) began talking. The first comment, below, refers to teacher J (another teacher in the group) who had mentioned at the start of the meeting that he was interested in promoting more 'independence' in the students he teaches.

> [Transcription conventions: //text// indicates overlapping speech; [text] is a transcriber comment; [2] indicates a pause of 2 seconds; other punctuation has been used to give some sense of phrasing; … indicates some text has been skipped, for ease of reading]
>
> P: I couldn't stop watching, thinking of you [P looks at J] and your independent children [Alf raises his hand towards P] and unfortunately all //the children that weren't paying attention//
>
> //Alf: So, so, so//
>
> //J: Yeah, yeah//
>
> Alf: That's an interpretation. So, at this stage, the invitation is to say what you saw, what you observed [1] so [1] how did it begin?

I remember feeling taken aback that P had begun talking before any invitation from me (in which I would usually have re-iterated the task of description and staying with the detail). On reflection, P's comment was, I suspect, extremely helpful to the group in terms of allowing me to give feedback in relation to a discussion norm, from the start. The distinction on offer here is that we cannot observe 'not paying attention'. What we might see is, say, children looking away or playing with items on a desk or talking – from which it is an interpretation that they are not paying attention.

Incident 2

My intervention, in Incident 1, did not ensure that conversation thereafter remained at the detail level (and nor would I have expected it to). After 3 minutes the following interchange occurred (G and J are commenting about a student on the video clip).

> G: He said that it wouldn't work if your one whole was 10?
>
> J: Yeah, I think he was talking more on the discrete nature of number, he was thinking about things being discrete
>
> Alf: So, try to avoid interpreting what you think he was saying [Alf laughs] try and stay with [1] so, what did you hear him say?

I recognise being attuned, when the task for the teachers is one of description, to any mention by a teacher of what might be going on in the mind of a student on the video. For me, these are the easiest comments to spot that are interpretations and not descriptions. We cannot observe what a student may or may not be thinking, by way of explanation of what they say. So, when J suggests a student was thinking about the discrete nature of number, I am not surprised to observe myself intervening and re-emphasising the discussion norm for this phase.

Incident 3

A little later in the meeting I do something similar to the first two Incidents, for a third time. P, in the transcript below, begins referring to a student on the video who she had heard talk about 'atoms' (on a number line).

> P: Someone started saying about atoms, didn't they.
>
> J: And that other lad saying about a really long number line. He was saying, if you had the longest number line in the world you could.
>
> T: I thought that was interesting because he kind of got it right, it's the same concept//
>
> Alf: //[Alf interrupts T] That // sounds like an interpretation //
>
> J: //Interpretation, yeah//
>
> Alf: Try and stay with the detail, we'll come on to that in a second. Let's try and see if we can get the chronology of things.

A little like comments about what a student might or might not be thinking, any comment evaluating the video, for example, as here whether a student is right or wrong, I recognise as indicating a move into judgment and interpretation. Again, I intervene, in this case interrupting T's contribution, and re-state the task as getting 'the chronology of things', i.e., *what* happened *when* during the clip.

In J's re-voicing of my comment 'Interpretation, yeah' there could be evidence of him beginning to recognise the distinction I am making between interpretations and descriptions. Following this third incident, there are no others where I notice a judgment and the teacher discussion remains at the level of detail and description (with some re-viewings of sections of video) before I shift to the next phase of asking for interpretations and analysis of what was seen.

Further reflections

The technique of starting with a reconstruction of events was designed precisely to move participants out of 'judging' and into a space where it is more likely teachers can learn and observe something new in the video clips they watch (Jaworski, 1990). This is not the end of the process of working on video and, following the 'reconstruction phase' there is then an invitation to move into an analysis of the video and a drawing out of implications for participants' own classrooms; however, the analysis phase is beyond the scope of this report, further details about the entire way of working can be found in Coles (2013, 2014).

There is evidence that a discussion norm (about starting off with description and not interpretation) has been established in the first meeting. After three interventions by me to flag up when discussion has moved to interpretation, no more are needed. It appears that discussion norms *can* become established quickly in a group, with a facilitator prepared to intervene and make the criteria for intervention explicit to the group, so that those criteria can become ones that participants are able to apply to themselves. There is no evidence of the cited difficulties of 'cognitive load' (Gaudin and Chalies, 2015, p.29) required for these teachers to work with video and adopt the way of working, even though several of them were newly qualified.

Reflecting on the way this first meeting went, there is a paradoxical sounding sense in which my own judgmental interpretation of the 'kind' of comment made by teachers supported them in moving away from their own judgmental interpretations of the video. The nature of this apparent paradox – that the facilitator appears to support participants moving away from judgment through the use of judgment, is not a phenomena I have found reported previously and it is explored in the next section.

LEVELS OF COMMUNICATION, LEARNING AND ERROR

The use of video recordings with teachers of mathematics sets up a context in which there is communication about the communications in the classroom. This report sets up a third level of communication (about communications of video that are about communications in the classroom). In order to help untangle the webs of connections involved and, in particular, the paradox mentioned above, there is a need to draw on theory beyond mathematics education. I have found it instructive to go back to Bateson's (1972) views on communication, errors and learning, which were an early influence on me and which also form part of the background to enactivism.

Bateson (1972) distinguished three levels of learning to capture how animals (including humans) alter their behaviours over time. Learning 0 indicates the same response (at two different times) to the same stimulus (e.g., the bell rings and the dog salivates; I ask a student what is 7x8 and they answer correctly). Learning I indicates that between time 1 and time 2 there is a change in response to the same stimulus (e.g., the dog learns to salivate on hearing the bell; a student moves from not knowing 7x8 to being able to give the correct answer). Learning II indicates a change in the *way* Learning I takes place (e.g., a dog becomes more efficient than it was at learning in the context of Pavlov-style experiments; a student moves from memorising discrete multiplication facts, to being able to use commutativity and 'doubling' of known facts to derive others). Learning II is only observed in animals able to engage in communication about communication and cannot be taught directly since it cannot be specified by particular behaviours. Learning II concerns *how* we learn new behaviours, not the behaviours themselves.

These considerations indicate there are two kinds of error an organism can commit, where 'error' is interpreted as an action that is not well adapted to the context:

> The organism may use correctly the information which tells him (sic) from what set of alternatives he should choose, but choose the wrong alternative within this set; or
>
> He may choose from the wrong set of alternatives. (1972, p.291)

Errors of the first kind, if corrected, can lead to Learning I; over time I may memorise that 7x8 is 56, not another number (my errors were from choosing the wrong alternative within a set). Errors of the second kind can lead to Learning II; over time I may learn that I can work out 7x8 not just by trying to remember it (and often committing the first type of error) but, say, from knowing 7x2 and doubling twice. My previous errors now can be seen as coming from trying to do the wrong *kind* of thing (for me) in memorising, compared to building on what I know (my errors were from choosing the wrong set of alternatives). NB What counts as 'wrong' in the example above, and in any instance, is relative to individuals and context.

When teachers speak judgmentally in the first phase of video watching, my feedback to them indicates that they are making this second kind of error. I am not questioning the interest or validity of what they say, but what I feedback to them is that they have made an error in terms of the *kind* of thing they are saying – they have made a choice from the wrong set of alternatives and, described in this way, the paradox seems to dissipate. My judgments are at a different 'logical' level to the communications and judgments about the video and so do not conflict with them.

Making the shift that teachers show they have done, in the evidence above, is evidence of Learning II – they have made a shift in the *way* they go about learning from video. When working with teachers of mathematics on video my aim is, precisely, to support a new way of acting and seeing. I want to allow for the emergence of new descriptions and, with those new descriptions, a possibility for new actions (see Coles 2013, 2014). The intention is to provoke Learning II and it is perhaps no surprise that so many research projects report that learning from using video is hard to facilitate. The kind of learning we are aiming for *should* be hard because, as humans, we can get ingrained in the set of alternatives from which we choose, in any given context.

A website has been created with resources to support facilitators in using video (www.mathsvideoclubs.ac.uk). We hope in the future to track the influence of these video clubs on teachers' developing practice and the on going learning of facilitators.

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USING THE BIG IDEAS OF MATHEMATICS TO 'CLOSE THE GAP'

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Researchers at QUT have developed a mathematics pedagogy (AIM) that seeks to achieve deep learning of powerful mathematics, particularly in Indigenous and low SES schools. AIM is a vertically structured pedagogy intended to 'close the gap' for underperforming students. It is based on the big ideas of mathematics, drawing on the Piagetian notion of schemas and Skemp's approach to relational understanding of mathematics. After initially partnering with eight schools to provide teacher training and support for the implementation of AIM the program, early outcomes were encouraging. There is evidence to suggest that a program based on big ideas of mathematics and vertical sequencing enabled significant acceleration.

INTRODUCTION

The mathematics performance gap between Indigenous and low SES students and other students is particularly large in Australian schools. In response to this, researchers at the YuMi Deadly Centre (YDC) at the Queensland University of Technology (QUT) developed a mathematics pedagogy called YuMi Deadly Maths (YDM). YDM seeks to achieve deep learning of powerful mathematics, particularly in Indigenous and low SES schools, so that students have improved employment and life chances. YDM embraces the 'big ideas' of mathematics, that is, ideas that illuminate a variety of topics across many year levels, as a central framework for the teaching of mathematics in the primary and secondary years. It is the basis of several different programs that assist mathematics teachers in addressing the needs of students at all levels of mathematical understanding,

This paper focuses on a YDM program called Accelerated Inclusive Mathematics (AIM) that uses big ideas and a vertical curriculum to accelerate learning of junior secondary Indigenous and low SES students to 'close the gap' between these students and other students. The paper is an initial analysis of the effect of the first interventions based on AIM on students in Indigenous and low SES schools in Queensland Australia. It discusses AIM's views on connections, schema, big ideas, sequencing and vertical curriculum, describes the design of the interventions, provides findings from the first case studies, and draws conclusions for the use of big ideas in remediation and acceleration. It complements a more theoretical paper on the definition and use of the big ideas of mathematics in YDM by Carter, Cooper and Lowe (2016).

THEORIES UNDERLYING AIM

AIM is a remedial pedagogy that is based on accelerated unlearning/relearning of mathematics ideas. It is based on constructivist theories that come from the work of

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Piaget (1977) and Vygotsky (1978) that individuals need to actively construct meaning from experiencing guided activities in the social milieu (Davydov, 1995; Jardine, 2006). It is also based on the importance of this meaning relating to the fundamental structures of mathematical knowledge, particularly the importance of students' knowledge being relational (Skemp, 1977), principled/conceptual (Leinhardt, 1990) and structural (Sfard, 1991). It integrates constructivism and structural knowledge based on the principles of Alexander and Murphy (1998) and taking account of the integrity of structure alluded to in English (2007) and English and Sriraman (2010).

Connections, schema and big ideas. AIM assumes that humans learn by organising knowledge into schemas that are stored for use when needed to understand and respond to situations. Learning occurs by increasing the number and complexity of the schemas and integrating schemas by adaptation (adjustment) to the world, through the processes of assimilation or accommodation (Piaget 1977). Where possible, existing schemas are used to understand (assimilate) new information. A general schema that is not context-specific can facilitate the assimilation of many types of new information (Richland, Stigler, & Holyoak, 2012). It is such schemas (called big ideas) that are the basis of AIM.

Successful assimilation creates a state of equilibrium in the learner. When the new information cannot be assimilated into existing schemas, a state of disequilibrium occurs which is resolved (schemas are changed or supplemented) through the process of accommodation. It follows that learning is easier if assimilation is possible, and this is more likely if big schemas (big ideas) are available. However, if learners fail to develop a relational understanding of mathematics (Skemp, 1976) as a framework of connected big ideas, there are few adequate schemas to draw on to assimilate new knowledge. The result can be a large number of disconnected facts that cannot be generalised and require drill and practice methods to ensure future recall (called instrumental knowledge by Skemp).

To reinforce underlying mathematical principles, there has been recent renewed interest in big ideas (Askew, 2013). This interest has resulted in big ideas being seen as the central organizing ideas (Schifter & Fosnot, 1993) that robustly link many mathematical understandings into a coherent whole (Charles, 2005). Big ideas have been characterised as having potential for: (a) encouraging learning with understanding of conceptual knowledge; (b) developing meta-knowledge about mathematics; supporting the ability to communicate meaningfully about mathematics; and (c) encouraging the design of rich learning opportunities that support students' learning processes (Kuntze et al., 2011). It has been argued that relating new concepts to big ideas promotes understanding, thus enhancing motivation, further understanding, memory, transfer, attitudes and beliefs, and autonomy of learning (Lambdin, 2003). Many argue, explicitly or implicitly, for the need for the big ideas to transcend the various branches of mathematics and also year levels (e.g., Morgan, 2012; Siemon, Bleckly, & Neal, 2012). However, as Carter et. al (2016) argued, agreement is harder to find when it comes to listing the big ideas.

AIM uses such ideas as an effective way of accelerating underperforming students. The program includes a taxonomy and detailed list of big ideas (see Carter et al., 2016).

Sequencing, vertical curriculum and big ideas. It is crucial that sequencing between connected ideas is seamless, where the transition from one idea to the next is not impeded by concepts taught in a way that do not support (or, worse, are contrary to) future developments. For example, if whole numbers are taught by adding like place values (and renaming if needed), the ground is prepared for algebraic addition which involves adding like variables. In contrast, denying the existence of negative values in the early years (for example, "you can't take 5 from 3") leads to confusion when subtraction requires regrouping.

AIM was designed for the junior secondary years where it provides teacher professional development (PD) and resources for use with students who are more than three years behind their age level in mathematics performance. They have been designed to teach six years of mathematics in three. To do this, a series of vertically sequenced modules covering Year 3 to 9 content were developed, each focussing on a few big ideas. They were based on the structured sequencing theory (Cooper & Warren, 2011) that was designed to develop big ideas across time. Figure 1 diagrammatically shows the difference between AIM's vertical curriculum and traditional horizontal curriculum. The horizontal approach, more commonly used to teach school mathematics, in which every topic is taught each year, is shown at the left of Figure 1. AIM's vertical structure, which teaches one-third of the topics in each year, arranged so that by the end of three years, all topics in the curriculum have been covered, is shown at the right of Figure 1. Both approaches seek to reach the same outcome by the end of Year 9 but in different ways.

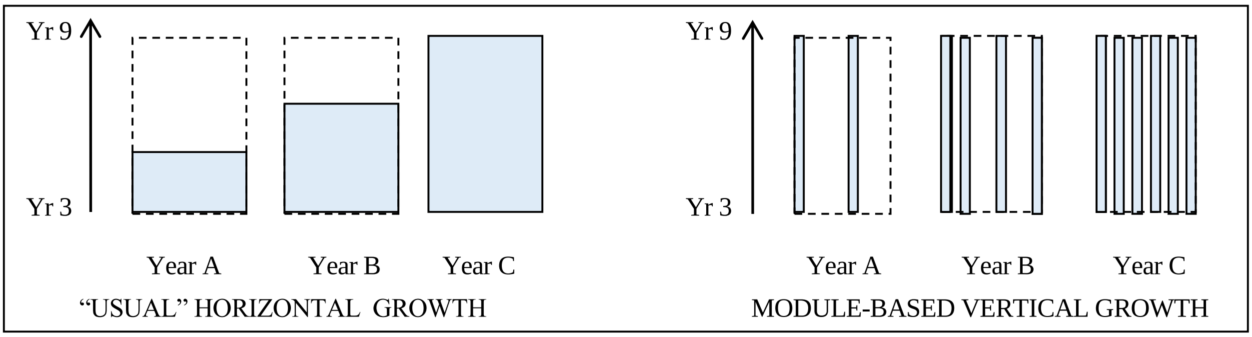

Figure 1. Normal/horizontal and module-based/vertical mathematics growth

Horizontal programs that revisit topics iteratively (often more than annually) usually take a spiralling approach. Each time a topic is revisited, there is a review of past learning to refresh the students' knowledge and provide links to the proposed new learning, followed by new work that builds on additional layers of knowledge and complexity. A vertical program based on modules, each of which develops a big idea from the foundations to advanced concepts, can be more time-efficient by eliminating the need to regularly revisit past learning, and because a good foundational knowledge of a big idea accelerates its future application. However, the use of modules results in only a small number of topics being taught in a particular year, with each of these topics are taught up to the Year 9 level. In other words, growth in mathematics changes from enhancing learning of all topics across each of the three years to adding new topics to fill in the gaps.

DESIGN OF INTERVENTIONS

YDC achieved its mission by entering into partnerships with schools to provide training and support for the implementation of the program in the school. Teachers attended PD for six to eight days per year for two or three years, depending on project arrangements between YDC and client schools. The schools also received resources and support in the form of an AIM Overview book, 24 half-term teaching modules (eight per year) each with pre-post tests, an online website, school visits by YDC practitioners if required, and a coordinator to organise the PD and answer questions. The training focussed on the AIM pedagogy and how to implement it in schools, and encouraged the teachers to trial the modules using an action-research approach.

The school AIM projects were design-based case studies (Cobb, Confrey, Lehrer & Schauble, 2003) using mixed methods data gathering techniques (observations, interviews, surveys, observations, teachers' feedback, and class pre-post test responses). They were also based on the empowering outcomes decolonising methodology (Smith, 1999), with the research designed to benefit the researched. The trained teachers were encouraged to trial the AIM ideas in their classrooms and then undertake in-school training of other teachers. For the trials, the trained teachers were asked to give the pre-post tests in the modules to their students and provide de-identified class responses to YDC staff. The data was considered using a case study approach. Within each case, the data was analysed to determine any changes in students' mathematics knowledge and engagement. These changes were related to the characteristics of the case, PD activity, the trained teachers' responses about teaching and training other teachers, and other teachers' responses about teaching (where relevant), in order to determine the reasons for them.

As discussed earlier, the goal of the AIM project was to develop, over three years, the mathematical knowledge and skills that would usually be covered in the first ten years of schooling. To achieve this, the AIM program provided 24 vertically sequenced half-term modules based on big ideas that covered all mathematics up to Year 9. They were divided into three years, covering basics, multiplicative ideas, and generalisation, respectively.

Like all YDM programs, AIM did not seek to provide a prescriptive 'of the shelf' teaching recipe. Instead, it aimed to expand teacher capacity through a multi-year program of training and support that included pedagogical approaches, mathematical content, teaching ideas, and activities, structured around the big ideas of mathematics. However, teachers were encouraged to make their own pedagogical decisions based on understanding of big ideas and knowledge of their students.

RESULTS AND DISCUSSION OF FIRST INTERVENTIONS

The first interventions were trials of the eight first-year modules. They were (in order) whole numbers, decimal numbers, addition and subtraction, length-capacity-mass, multiplication and division, perimeter-area-volume, 2D and 3D shape, and tables and graphs. The first year was designed to prepare students for vocational education and training (VET) options (e.g., Certificate II in a trade). The module resources provided active teaching ideas that started from the context/culture of the students, moved through body→hand→mind activities and reflected back to the students' world. The interventions were undertaken with nine secondary schools, summarised in Table 1. Six of the schools had more than 95% Indigenous students. Each school had at least one Year 8 class with greater than 25% Indigenous students in which all students were operating at mid-primary level in mathematics (there were 14 classes in all). At the time, Queensland secondary schools commenced at Year 8 (and not Year 7 as now).

Table 1. Schools in AIM trials

School	System	School type	Remote	Over 95% Indigenous
1	Independent	Boarding/Day	No	Yes
2	Independent	Boarding	No	Yes
3	State	Community school	Yes-culturally	Yes
4	Catholic	Boarding	No	Yes
5	Catholic	Boarding/Day	No	No – 25%
6	Catholic	Boarding/Day	No	No – 25%
7	Independent	Community school	Yes-culturally	Yes
8	State	Boarding/Day	Yes-distance	No – 25%
9	State	Community school	Yes-culturally	Yes

The intervention results were affected by the schools' characteristics, including (a) challenges in terms of behaviour and attendance, (b) rapid turnover of principals, teachers and students (particularly for the community schools), (c) continual changes from systems in the nature of the mathematics pedagogy, and (d) community activities changing school activity. The findings were also affected by the teachers' characteristics such as (a) extent of change in teaching to meet the needs of AIM; (b) acceptance of and use of big ideas; (c) knowledge of mathematics and mathematics education (over 80% of the teachers in the AIM program were teaching out of field and had no tertiary training in mathematics or mathematics teaching); and (d) readiness to spend time ensuring pre-post tests were administered correctly. This paper focuses on four cases where the outcomes were relevant to big ideas pedagogical approach.

Decimal results higher than whole numbers. One of the startling initial results was that, although pre-tests showed that students understood whole numbers better than decimals, the pre-post tests showed greater improvement in decimals, resulting in students' knowledge of decimal numbers that was significantly better than of whole numbers. In interviews, teachers agreed with this result, explaining that it was because the whole number work helped decimal number understanding. The whole and decimal number modules were built around the same five big ideas: notion of unit and part-whole; additive structure and counting; multiplicative relationships; number line; and equivalence. Teachers reported that their students stated that the decimal work was "the same" as for the whole number, showing that they recognised the structural similarity. Thus, the whole number modules prepared the students for the decimal module and the organic nature of big ideas (Skemp, 1976) enabled assimilation and acceleration of learning.

Additionally, AIM ensured that all learning in the whole numbers module was seamlessly sequenced with decimal numbers. For example, the multiplicative structure presented multiplying or dividing by 10 as the left or right movement of place-value positions (not adding and removing zeros). The extra work that this required in whole numbers was compensated by the acceleration of learning in decimals. This result was a strong validation for the AIM approach to acceleration and result confirmed the vertical curriculum as the AIM structure.

Multiplication/division results not higher than addition/subtraction. Whilst the addition/subtraction and multiplication/division modules were also based on the same big ideas, there was not the same improvement in the later module, seemingly contradicting the whole number/decimal findings. In discussions, the teachers admitted to not following the big ideas approach of the modules. This was supported by PD leaders who advised that the teachers found these modules on operations too difficult and felt they could not teach them. The teachers considered that the modules contained too many different big ideas in which they lacked experience. The operations had been explained in terms of the big ideas of meaning (concepts), relationships/laws (principles) and separating into parts (strategies), but the teachers saw operations only in terms of computation. In the second year, the principle big ideas were moved to a new third year module on translating arithmetic principles to algebra. With only concepts and strategies to deal with, which could be related to computation, the teachers came to believe that they could teach the module and problems disappeared, demonstrating that program knowledge cannot outpace teacher knowledge.

Enhanced capacity of teachers, including out-of-field teachers. There were significant increases in the teachers' capacity to teach mathematics where there was staff stability. Teachers came to like teaching with AIM; becoming more motivated, confident and knowledgeable about mathematics and its teaching over time. In particular, this was true for out-of-field teachers, a finding supported by a Queensland Government audit of teachers in Queensland. The audit highlighted AIM as an example of a PD program that trained out-of-field teachers to be able to teach mathematics effectively. This was particularly important because AIM's target schools have high numbers of out-of-field teachers (over 80% in this AIM intervention). AIM's focus on big ideas and teacher capacity had a two-fold positive effect, on teachers as well as students.

Student participation in senior secondary improved. AIM pre-post test results were nearly always strongly positive for all modules, but the main objective of AIM was that students would be able to access senior mathematics subjects after three years. In the past, mathematically underperforming students dropped out or failed in Year 10 and left school with low employability. However, this trend was reversed in those AIM schools that had stability in leadership, staff and students. Principals reported that students were continuing into Year 11 with good mathematics performance.

One boarding school had the strongest success, with 13 of the 16 students who started AIM in Year 8 succeeding in University entrance mathematics courses in Year 11. This school taught AIM as a support to their normal mathematics subjects, using the resources to develop their own programs. Their teacher said that AIM had given the students confidence in their ability to continue studying mathematics. Teachers using the modules as a resource to improve their existing programs has become a major part of the latest AIM projects, with recent reports that AIM students are outperforming their non-AIM counterparts who were assessed as having higher initial knowledge.

CONCLUSIONS

These are only a small indication of findings of the analysis of AIM's effects on underperforming students. Although effects on Indigenous and low SES schools are difficult to unpack, there is a strong belief that a mathematics programs based on big ideas and vertical sequencing enabled significant acceleration. The program must take account of teacher knowledge and student knowledge, but can provide two-fold outcomes. The evidence suggests that a big ideas approach works because it: (a) covers many mathematical ideas; (b) reduces need for rote procedures; and (c) is organic allowing later work to be assimilated. It also argues that vertical sequences work because the same ideas operate throughout the module and early learning provides the foundations for later learning. It also seems that big ideas and vertical sequences are particularly suited to students from Australian Indigenous and low SES backgrounds, the main targets of AIM. These learners tend to be holistic in learning style, moving from whole to parts, and not aligned with the traditional algorithmic teaching methods that move from parts to whole.

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RECIPROCAL RELATIONS OF RELATIVE SIZE IN THE INSTRUCTIONAL CONTEXT OF FRACTIONS AS MEASURES

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The presented study is part of a bigger design and research enterprise in the teaching of fractions as measures. We analyze extracts of a teaching session with a single fifth grade student, in which he flexibly compared the relative sizes of the lengths of three drinking straws, skillfully using unitary, proper, and improper fractions. We identify aspects of his prior instructional experiences that supported the emergence of his relatively sophisticated ways of reasoning. Findings suggest that supporting students' reasoning about reciprocal relations of relative size can be a viable goal in an instructional agenda on fractions as measures.

In this paper, we explore how to instructionally support the emergence of a form of mathematical reasoning that involves reciprocally quantifying the size of two magnitude values (Ramful, 2013; Thompson & Saldanha, 2003). Quantifying in this way is central to many scientific and everyday practices. For instance, in money exchange, the relative value of two currencies is determined in a reciprocal way, so that if the value of an Australian dollar is 16 Mexican pesos, the value of a Mexican peso is 1/16 of an Australian dollar (or 0.0625 AUD).

A key aspect of reciprocal quantitative comparisons is that they always involve the use of rational numbers. In the example above, although the value of an Australian dollar relative to a Mexican peso can be quantified using only natural numbers, quantifying the reciprocal relation necessarily requires the use of fractions (or an equivalent rational expression). The relatively sophisticated fraction understanding that is required to conduct this kind of comparisons, makes them difficult to study in elementary years, where such levels of fraction understanding are seldom developed (e.g., Hannula, 2003).

We analyze the reasoning of a fifth grade student, Pedro, who became skillful in using fractions to determine the relative size of up to three different magnitude values in a reciprocal way. In the analysis, we also identify aspects of Pedro's reasoning that can be directly linked to specific instructional experiences. Based on the results of the analysis, we then consider the resources for classroom teaching that would need to be developed so that proficient teachers could support the development of similar ways of reasoning in regular classrooms.

BACKGROUND

Pedro attended an urban public school in Mexico, and was experiencing difficulties with learning mathematics. The first author worked with him, after school, in one hour weekly sessions, both to help him overcome his difficulties, and as part of a broader

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research effort (Cortina, Visnovska, & Zúñiga, 2014a). This effort has centered on developing an instructional sequence on fractions as measures (Kieren, 1980), using the general methodological approach of *design research* (Gravemeijer & Cobb, 2006).

In our instructional design, we follow the tenets of Realistic Mathematics Education (Gravemeijer, 1994), as well as the adaptations of this theory proposed by Cobb and colleagues (Cobb, Zhao, & Visnovska, 2008). We aim to (a) clarify the progression of forms of student reasoning that are likely to emerge as students engage in specific instructional activities, and (b) provide guidance for the teacher about how the emergence of these forms of reasoning can be proactively supported in classrooms.

It is worth clarifying that we have developed the instructional sequence by working with groups of students in their regular classrooms. The first author's involvement in Pedro's remedial education was used as an opportunity to explore the continuation of the sequence, prior to its classroom testing. In this sense, the presented analysis belongs to the 'preparing for a classroom experiment' phase of design research and will be used to formulate the rationale of the extended sequence.

The samples of Pedro's reasoning that we analyze in this paper come from the 21st weekly remedial session. In sessions 8-20, the first author guided Pedro through a previously developed sequence of instructional activities on fractions as measures (Cortina et al., 2014a). The progression of Pedro's reasoning during these weeks was consistent with prior findings. We include the overview of these developments to clarify the instructional approach taken, particularly with respect to how Pedro was supported to understand unitary and common fractions. Against this background we then introduce the activity of week 21, in which Pedro engaged in instructional activity that involved making reciprocal quantitative comparisons.

SUPPORTING PEDRO TO REASON ABOUT FRACTIONS AS MEASURES

Pedro was first asked to use parts of his body (e.g., hand span) to measure lengths and reflected on the advantages and disadvantages of gauging the lengths of objects in this way. He was then asked to measure lengths using a standardized unit, a wooden stick with no marks on it, about 24 cm long. In doing so, the issue of the remainder became a concern, as he realized that many objects did not measure a whole number of iterations of the stick. For instance, a table would measure three sticks and a bit more.

In order to quantify the lengths of the remainders, Pedro was oriented to produce subunits of measure, the lengths of which corresponded to unit fractions of the length of the stick. Importantly, he was not led to construe unit fractions as quotients of a partitive division (i.e., the result of equally partitioning a whole into a certain number of equal size parts). Instead, he was supported to construe them as divisors in a measurement division, in which the reference unit is divided a whole number of times, with no remainder.

To clarify the distinction we are making, asking a student to fold a plastic drinking straw of the length equal to a stick, twice, and think about the length of the resulting four equal size parts, would correspond to construing 1/4 as a quotient of a partitive division. In contrast, Pedro was given a plastic drinking straw shorter than the stick and asked to cut it (i.e., to create the divisor) so that when using it to measure the stick, it would fit in exactly four times (see Figure 1). He did this through a process of trial and error in which he would cut the straw and test it to see if its length met the specified condition. If the straw turned out to be too long, he would cut it shorter. If it turned out to be too short, he would start afresh with a new straw. We believe that the distinction described made the difference to Pedro's reasoning and return to it in the discussion section of this paper.

Figure 1: 1/4 as the length of a subunit (a plastic drinking straw) that fits exactly four times in the length of a reference unit (a white wooden stick).

Using the described procedure, Pedro produced straws of the following lengths: 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9 and 1/10, in this order. Consistent with findings from prior experiments (Cortina et al., 2014a), by reflecting on the process of producing unit fractions in this way, Pedro came to develop a comprehensive understanding of the inverse order relation of unit fractions (i.e., the bigger the number in the denominator, the smaller the fraction size).

Next, Pedro was oriented to interpret common fractions as measures that accounted for a length that corresponded to the iteration of a subunit a certain number of times. For instance, the fraction 7/4 would account for a length that corresponded to seven iterations of a subunit of length 1/4.

Pedro was then supported to recognize how subunits, when iterated a specific whole number of times, rendered a length identical to that of the reference unit (e.g., $4 \times 1/4 = 1$). Also consistent with prior findings, developing such an understanding allowed him to correctly judge any fraction as representing a length smaller than, as big as, or bigger than one (e.g., $2/3 < 1$; $3/3 = 1$; $4/3 > 1$). It also allowed him to soundly and correctly convert improper fractions into mixed fractions and vice versa.

When Pedro began to engage with the instructional activities that had not been previously tested in classrooms, the first author took photographs of his work, created detailed field notes after each teaching session, and debriefed Pedro's learning with the second author weekly. The first problem related to assessing reciprocal relations was presented to Pedro in session 20, and his relatively seamless response motivated video recording of the subsequent session, which we analyze here.

REASONING ABOUT RECIPOCAL RELATIONS OF RELATIVE SIZE

The 21st session started by asking Pedro to cut a small straw of about the same length as his little finger. This straw was initially construed as having the measure *one*. He was then asked to produce two more straws, one that measured four (small) straws, and another seven. Pedro chose to label the shortest straw as "*A*", the middle one as "*B*", and the longest one as "*C*" (see Figure 2).

The teacher then asked Pedro to decide which of the two longer straws would now play the role of being "the stick" (i.e., the reference unit) and, thus, being attributed the measure *one*. The conversation below (translated from Spanish) then took place, where P and T refer to Pedro and teacher respectively.

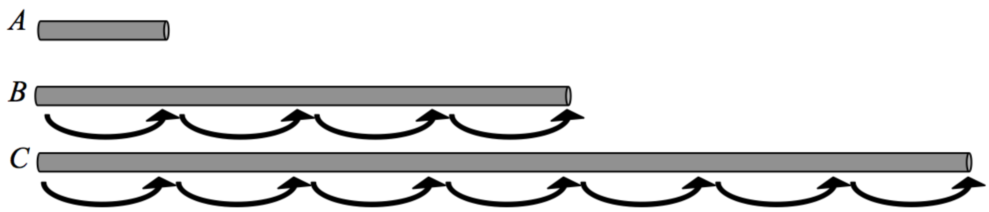

Figure 2: Straws *A*, *B* (4 times as long as *A*), and *C* (seven times as long as *A*).

A1	P:	*B* will be one so *C* will be two (chuckling)?
A2	T:	Let's focus on *A* first.
A3	P:	*A* will be two.
A4	T:	Let's see, why two?
A5	P:	No, *A* is going to be four (showing four fingers).
A6	T:	And what is bigger, one or four?
A7	P:	Four.
A8	T:	So is this longer than this (placing straw *A* next to straw *B*).
A9	P:	No. (Pause). Then it would be smaller? No? (Looking at the teacher).
A10	T:	Let's see. If this is your stick (pointing at straw *B*), what is this (holding straw *A*)?
A11	P:	A fourth.
A12	T:	Ok. Why a fourth?
A13	P:	Because *B* is divided into four (gesturing with his hand along straw *B*), and since I have a fourth, then it is one, two, three, four (taking straw *A* and iterating it along straw *B* as he counted).
A14	T:	Ok, write it in the table (Pedro's record of reciprocal comparisons).

This first extract illustrates how Pedro came to reason quantitatively about the reciprocal of iterating the length of a unit, a whole number of times. With relatively little support from the teacher (see utterance A10), he seemed to have readily reinterpreted the situation as one in which the length of a subunit was to be quantified. He now interpreted the length of straw *A* in the way he had been oriented to construe the quantitative meaning of a unit fraction—in this case *one fourth*—as being the length of a subunit that would fit four times, exactly, in the length of the reference unit (see utterance A13 and Figure 1). Evidently, his reasoning was consistent with his prior instructional experiences in the remedial teaching sessions.

Pedro was next asked to determine the length of *C*, relative to the length of *B*.

B1	P:	Then, *C* would be bigger than *B* (looking at straws *B* and *C*)
B2	T:	Ok.
B3	P:	Four (likely meaning the length of straw *B*), they would be (touching straw *C*, closing his eyes and pausing to think) a seventh?
B4	T:	A seventh? What is bigger, a whole (referring to straw B) or a seventh (referring to straw C)?
B5	P:	A whole.
B6	T:	So this one (touching straw *B*) is longer than this one (touching straw *C*)?
B7	P:	Oh, no.
B8	T:	So how can *C* be a seventh of *B?*
B9	P:	Oh no. Then it would be one whole (closing one eye and pausing to think) three (short pause) fourths?
B10	T:	Why?
B11	P:	Because there are four here (gesturing with his hand along straw *B*), and there are four here (gesturing with his hand in the same way along part of straw *C*), but there are three more here (pointing at the rest of the length of straw *C*), so a whole has been formed, with three fourths (added to it).

This second extract illustrates how Pedro reasoned about the size of a length (*C*) relative to size of a new reference unit (*B*). He initially seemed to focus on both the lengths of straws *B* and *C* as being a product of iterating the length of straw *A* a certain number of times. This might have led him to think first about the inverse of producing a length seven times as long as *A* (i.e., a seventh). However, prompted by the realization that *C* could not be one seventh of *B*, he seemed to have then realized that straw *C*, by being the product of iterating *A* seven times, would be three iterations of one fourth of *B* longer than *B*. Pedro seemed to have relied on his prior realization that the length of straw *A* was one fourth of the length of *B*, and on the fact that *C* was three iterations of *A* longer than *B* (see utterance B11).

Here too, Pedro's reasoning was consistent with his instructional history in the remedial teaching sessions. More specifically, it was consistent with how he had been supported to interpret the meaning of common fractions in the teaching sessions—as lengths produced by iterating a subunit a certain number of times.

Finally in this problem, Pedro was asked to determine the lengths of *A* and *B* relative to the length of *C*.

C1	P:	Then, if *C* is one, it (meaning *A*) would be (pause) one seventh?
C2	T:	Are you just guessing?
C3	P:	No. That one would actually be a seventh (pointing at *A*).
C4	T:	A seventh, why?
C5	P:	Because it fits seven times in *C*. And *B* would have four sevenths.
C6	T:	Ok. Why?

C7 P: Because here (aligning straws *B* and *C*) if you measure it (meaning "with *A*"), there are four here (touching the *B* straw) and seven here (touching the *C* straw). But if you join them, there are four sevenths here (touching the *B* straw). So it is four sevenths.

In this final interaction, Pedro engaged in the same kind of reasoning as earlier. He seemed to have readily reconceptualised *C* from being a length seven times as long as *A*, to being a reference unit into which the subunit *A* would fit seven times; and *B*, from being of a length four times as long as *A*, to being a straw as long as four iterations of a seventh of the length of straw *C* (see utterance C7).

In the remainder of the session, Pedro engaged in similar reasoning with relative ease. He correctly compared the lengths of three other straws (1, 3, and 10) without physically creating them. Afterwards, he succeeded in establishing that his age (10 years) was 10/13 of his sister's, and his sister's age was his plus 3/10 of his age. Finally, he determined the fraction of the student population of his school, in his classroom (32/407), as well as the size of the school population relative to the number of students in his classroom (407/32).

CONCLUSION AND DISCUSION

The analysis of Pedro's reasoning provides a justification for extending the instructional sequence on fractions as measures, towards a goal of supporting students' reasoning about reciprocal relations of relative size. As the findings indicate, Pedro's relatively sophisticated ways of reasoning were tightly linked to his instructional experiences, and reflected how unitary and common fractions were conceptualised within the sequence. It is reasonable to expect that when using this sequence in a classroom, some of the students' reasoning elicited by the reciprocal comparison tasks would be similar to Pedro's. According to the theory of Realistic Mathematics Education, a teacher could then aim to advance the instructional agenda by making such reasoning the focus of collective analysis and discussion, while proactively supporting the sense making of all students.

A classroom design experiment is required not only to trial this process, but to design appropriate resources for classroom teaching on which a teacher could build. Such resources would include (but are not limited to) accounts of the diversity of student reasoning in specific instructional activities, and symbolic and other means of supporting mathematical conversations in the classroom.

Regarding a broader agenda on the teaching of fractions, the analysis illustrates the developmental advantages of supporting students to make sense of the quantitative meaning of unitary fractions, primarily as divisors in a measurement division. In a recent paper, Beckmann and Izsák (2015) propose a distinction between two quantitatively different ways of construing ratios in instruction, which they convincingly argue can have significant implications for how students come to understand this idea. The distinction they make is closely related to the difference between the measurement and partitive meanings of division, as well as between two meanings for multiplication tightly linked to those two of division. We believe that a similar distinction could be important to consider in fractions instruction.

In terms of multiplicative relationships (Beckmann & Izsák, 2015), our analysis shows that central to Pedro's success in reasoning about reciprocal relations of relative size was his reconceptualization of *the iteration* of a given unquantified magnitude value A, from being a multiplication ($n \times A = B$) to being a measurement division ($B \div A = n$). Such a reconceptualization entailed the reinterpretation of the produced magnitude value B, from being a product n times as big as the original reference unit (when $A=1$, $B=n\times1$), to being a reference unit in its own right ($B=1$). In addition, it entailed reinterpreting the original magnitude value A, from being a multiplied reference unit (i.e., a multiplicand in $B=n\times A$), to being a divisor, of a measurement division, that would divide the new reference unit, B, n-times exactly ($1 \div A = n$), and would thus be one n-th as big as the new reference unit ($A=1/n \times 1$).

Despite the apparent complexity of the reasoning just described, it seems to have been readily available to Pedro. For us, this is unsurprising given his prior instructional experiences in the remedial teaching sessions. As we explain above, the image of a unit fraction Pedro was purposefully supported to develop was not the typical one used in initial fraction instruction—namely, that of the size of a part of an equally partitioned whole (Cortina et al., 2014a). Instead, it was that of the length of an object (a plastic drinking straw) that would exactly fit a whole number of times into the length of a rather arbitrarily defined reference unit of measure (a wooden stick; see Figure 1). Such an image is consistent with regarding a unit fraction as a divisor, in a measurement division, that divides a reference unit a whole number of times, with no remainder. Once Pedro conceived the originally iterated magnitude value A as having the value of $1/n$, it seems to have also been rather easy for him to reinterpret the iterations of this magnitude value as being iterations of a unit fraction. Thus, he could then soundly construe the iterations of the original magnitude value as iterations of one n-th of a new reference unit.

The specialized literature is abundant with descriptions of how students struggle to make sense of fraction-related ideas, as well as with evidence that shows that very few children develop sound understandings of fraction notions that they are expected to master during their elementary education. By and large, fractions have been portrayed in the literature as conceptually complex, and making sense of them as quite difficult for most students. Elsewhere (Cortina, Visnovska, & Zúñiga, 2014b) we have advanced the conjecture that much of the documented difficulties in fraction learning can be regarded as a function of the trajectories that instructional designers, teachers, and researchers have expected students to follow in learning fractions. More specifically, we have conjectured that those difficulties can be a function of expecting students to develop an initial and basic quantitative meaning of a unit fraction as the size of a part of an equally partitioned whole.

Pedro's case illustrates how helping students to make sense of unitary fractions in an alternative way can make a difference. For Pedro, this alternative was very helpful when he came to reason quantitatively about the complex and challenging notion (Ramful, 2013) of reciprocal relations of relative size. Whether the proposed instructional sequence on fractions as measures would result in trajectories for students' fractions learning that do not run into the detours of traditional difficulties is the focus of our ongoing research agenda.

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THE TEACHER'S ROLE IN PROMOTING STUDENTS' RATIONALITY IN THE USE OF ALGEBRA AS A THINKING TOOL

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This paper deals with the crucial issue of the first approach to algebra as a thinking tool. A relevant excerpt from a teaching experiment is analysed through the use of two complementary theoretical tools: Habermas' concept of rational behaviour and the construct of Model of aware and effective attitudes and behaviours ($M_{AE}AB$). This analysis is carried out with the aim of highlighting how the different roles played by the teacher during class discussions promote students' rational behaviour.

INTRODUCTION

Many research studies point out that algebraic language should be presented and treated in classroom as a tool for representing, exploring relationships, interpreting and developing reasoning (see, as paradigmatic example, Arcavi, 1994). In tune with these research studies, both the authors have investigated the design and implementation of activities of proof construction through algebraic language (Cusi & Malara, 2009; Morselli & Boero, 2011) aimed at promoting algebra as a tool for thinking (Arzarello, Bazzini & Chiappini, 2001).

Few studies have focused on the role played by teacher's actions and interventions in fostering an effective and aware development of reasoning by algebraic language and on the interrelations between these roles and the thinking processes developed by the students. In this paper we will try to address these issues, integrating two theoretical tools (the construct of $M_{AE}AB$ and Habermas' construct of rational behaviour) in the analysis of a class discussion from a teaching experiment performed in grade 9.

THEORETICAL TOOLS

The $M_{AE}AB$ (acronym for Model of Aware and Effective Attitudes and Behaviours) theoretical construct is the result of a study aimed at highlighting the delicate role played by the teacher in effectively guiding his/her students to the construction of reasoning through algebraic language. It has been conceived within a Vygotskyan frame to the study of teaching-learning processes (Vygotsky, 1978) and takes into account the fundamental aspects that are connected to students' development of reasoning through algebraic language. A set of roles (summarised in the following table) have been identified (Cusi & Malara, 2009, 2013) to outline the approach of a teacher who consciously behave constantly aiming at "making thinking visible" (Collins et al., 1989), in order to make his/her students focus not only on syntactical or interpretative aspects, but also on the effective strategies adopted during the activity and on the meta-reflections on the actions which are performed.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 187–194. Szeged, Hungary: PME.

A first group of roles are those performed when the teacher tries to carry out the class activities posing him/herself not as a "mere expert" who proposes effective approaches, but as a learner who faces problems with the main aim of making the hidden thinking visible, highlighting the objectives, the meaning of the strategies and the interpretation of results.	*Investigating subject and constituent part of the class in the research work being activated*: when the teacher asks students to give suggestions about how to go on with the activity, intervening with the aim of making them feel involved in the activity as a group;
	Practical/ Strategic guide: when the teacher poses herself, in front of the problem, as an inquirer who aims at sharing the thinking processes and discussing the possible strategies to be activated;
	"Activator" of interpretative processes: when the teacher makes the students activated proper conceptual frames (Arzarello, Bazzini & Chiappini, 2001) to interpret the different algebraic expressions constructed when solving a problem;
	"Activator" of anticipating thoughts (Boero, 2001): when the teacher makes the objectives of the manipulation of algebraic expressions explicit and recall them during the discussion, in order to enable the students to share these objectives, monitor and control the activated strategies;
The second group of roles refers to the phases during which the teacher becomes also a point of reference for students, to help them clarify salient aspects at different levels, with an explicit connection to the knowledge they have already developed.	*Guide in fostering a harmonized balance between the syntactical and the semantic level*: when the teacher makes the students focus on the importance of controlling both syntactical and interpretative aspects and she discusses possible problems arisen when the syntactical or the interpretative level is not controlled;
	Reflective guide: when, in front of a student who proposes an effective approach to the resolution of a problem, the teacher asks him/her to make his/her thinking processes explicit, or she repeats what has been said by the student stressing on the reasons subtended to his/her approach, or she asks to other students to interpret what he/she said;
	"Activator" of reflective attitudes and meta-cognitive acts: when the teacher poses meta-level questions aimed at making the students evaluate the effectiveness of a strategy and reflect on the effects of a choice that was made during the resolution process.

Table 1: Characterisation of the roles played by a teacher as a $M_{AE}AB$

The second theoretical tool to which we will refer in our analysis is Habermas' construct of rationality. Drawing from this construct, Morselli & Boero (2009) propose that the discursive practice of proving encompasses:

"- an epistemic aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning […];

- a teleological aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product;
- a communicative aspect: the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture". (p. 100)

When proving by means of algebraic language, epistemic rationality consists of modeling requirements, inherent in the correctness of algebraic formalizations and interpretation of algebraic expressions, and systemic requirements, inherent in the correctness of transformation (correct application of syntactic rules of transformation); teleological rationality consists of the conscious choice and management of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity; communicative rationality consists of the adherence to the community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions (Morselli & Boero, 2011). The student must combine the adherence to syntactical rules on one side, and the goal-oriented management of the processes of formalization, transformation and interpretation, on the other. Still related to teleological rationality, the student must be aware of the fact that proving by algebraic language means deriving from algebraic manipulation a new algebraic expression, whose interpretation gives new information concerning the truth of the statement. In order to foster students' awareness of this, two levels of argumentation are identified as relevant: the meta-level, concerning the constraints related to the three components of rational behaviour in proving, and the proof content level (Boero et al., 2010).

RESEARCH QUESTIONS AND RESEARCH METHODOLOGY

In the following, we present our analysis of an excerpt from a class discussion, which was chosen because of the variety of argumentations at meta-level that are developed and because of the crucial role that the teacher plays. The analysis is carried out referring the theoretical tools previously introduced: (a) the construct of rational behaviour is used to analyse the students' thinking processes during the discussion; (b) the $M_{AE}AB$ construct is used to analyse the roles the teacher plays to develop a meta-level discussion focused on the ways of using algebra as a thinking tool.

The aim of this twofold analysis is to study the interrelation between the teacher's interventions (and the subsequent roles she plays during the discussion) and the students manifested thinking processes. Specifically, we focus on the following research questions: (1) how does the teacher deal with meta-level argumentations developed during the discussion? (2) what are the links between the teacher's roles and students' rational behaviour?

AN EXCERPT FROM A CLASS DISCUSSION

The discussion we are going to analyse was carried out during a teaching experiment, developed by one of the authors (Cusi & Malara, 2009), where an innovative introductory path to proof in elementary number theory (grades 9-10) was designed and implemented with the aim of fostering an approach to teaching algebra with a focus on the control of meanings. The class-based work was articulated through small-groups activities, collective discussions and individual tests. The data being analysed were students' written productions and the transcripts of the audio-recordings of both small-groups and whole class activities.

In this paper we will base on the transcript of a classroom discussion in grade 9, focused on the following task: *The sum between one number and its square is always an even number. Is it true or false? Why?*

Different proofs could be constructed: (a) a proof in natural language, referring to implicit theorems; (b) a verbal-algebraic proof, drawing on the fact that the considered sum could be written as the product between a number and the consecutive one; (c) an algebraic proof, which requires to distinguish between two cases. Because of space limitations and since the main focus of the discussion is on the algebraic proof, we will analyse only the third one.

The algebraic proof of the statement requires to activate the following anticipating thought: "in order to show that the expression $n+n^2$ always represents an even number, it should be written as the product between 2 and a natural number". The need of constructing an expression that could be transformed in the product between 2 and a natural number fosters the activation of the frame "even/odd", distinguishing between two cases. If the number is even, the sum between it and its square could be written as: $2x+(2x)^2=2x+4x^2=2(x+2x^2)$. If the number is odd, the sum could be written as: $(2x+1)+(2x+1)^2=2x+1+4x^2+4x+1=2+6x+4x^2=2(1+3x+2x^2)$. In both cases, the activation of the anticipating thought "the expression should be written as 2 multiplied by something" guides the treatments to be carried out, suggesting to carry out processes of transformation with the aim of taking out 2.

Analysis of the excerpt

After having worked in small groups, the students are involved by the teacher (T) in the analysis of the different approaches adopted by the groups of students to prove the statement.

In the initial part of the discussion, the class agrees that the statement is true. Two groups of students (group A and group B) propose their justifications:

(1) Group A's justification: $5^2+5=30$; (2) Group B's justification: $x+x^2=2y$.

S (who belongs to group B) asks to comment about his group's answer.

(11) S: We have done the same mistake we did before (*he refers to a previous activity*) … we have re-written the exercise, but in algebraic language.

…

(14) T: S is saying that the problem is that we are only re-writing the statement, but we are not motivating why it is true… And what do you think about P's group proposal?

S's intervention gives the first occasion for argumentation at meta level. S is able to recognize what was wrong in their solution. He is aware of the fact that the final aim is not re-writing the thesis of the statement (*teleological r.*). T revoices S' intervention, with the aim of sharing this reflection with the other students (*"Activator" of reflective attitudes*). Using the pronoun "we" and asking the students to shift the focus on the other attempt of proof, T also acts as an *Investigating subject and constituent part of the class*.

(15) F: It is right … but it is only an example.

(16) T: Is it a justification?

(17) M: No!

(18) P: It is not generalised!

(19) T: It is not a justification because this example says that the statement is true in this case, but it could be possible to find an other number for which the statement …

(20) Chorus: ...is not true!

An argumentation at meta-level on the value of numerical examples (*epistemic r.*) is developed. M and P recognize the only use of numerical examples could not represent a proof because it lacks in generality (*epistemic r.*). T acts again as an *"Activator" of reflective attitudes* with the aim of making students assess and control the processes that are activated.

Later, G proposes the justification given by her group: "The square of an even number is always even, the square of an odd number is always odd.... So the sum is always even". T involves the class in the analysis of this verbal proof of the statement. They discuss about how this "verbal approach" could be translated into algebraic language. One student, Max, proposes to start from the symbolic representation of an odd number. Afterwards, An says that she did something similar to what has been proposed by Max. T invites An to the blackboard, where she writes:

$$2n+(2n)^2=2n+4n^2= 2(n+2n^2)$$

$$(2n+1)+(2n+1)^2=2n+1+4n^2+4n+1=6n+2+4n^2= 2(3n+1+2n^2)$$

The proof proposed by An, complete and correct, takes into account both cases, highlighting how An effectively worked at the *epistemic* level. Moreover, the formalization and transformations are correct and possibly driven by the final aim (to find out divisibility by 2), and therefore highlighting that An also worked at the *teleological* level. This is a good occasion for another meta-level argumentation on the way of dealing with algebra as a proving tool. Then T involves the class in the analysis of An's proof:

(85) T: Is there someone who wants to explain what An has written on the blackboard?

(86) Al: She calculated the expression! *E raises her hand.*

(87) T: E, do you want to say something?

(88) E: She has separated the two cases: the first time with even numbers, the second time with odd numbers … and the results should be … (*hesitating*)

(89) T: And the result should always be … ?

With the aim of making the students focus on the meaning of the expressions constructed by An and on the objectives of the transformations she performed (85), T acts as both an *"Activator of Interpretative Processes"* and as an "*Activator of anticipating Thoughts*". E reacts to the teacher question (88) pointing out the final aim ("the result should be…"), highlighting *teleological rationality*. This is in contrast with Al's intervention (86), who seems not to have caught the final aim, and therefore highlight a *lack in teleological rationality*. T revoices the final part of E's intervention, focusing on the objectives of the transformation An has performed ("*Activator of anticipating Thoughts*").

(90) E: An even number! …

(91) Chorus: Even!

(92) E: So she has proved the thesis!

(93) T: An, why did you distinguish between even and odd?

(94) An: Because when we tried to use x and x^2 we were not able to prove it.

(95) T: An is saying "I have tried to write $x+x^2$, but I was not able to show that this sum is 2 multiplied by something". So she tried to distinguish between these two cases. Attention! We are considering two cases, so the proof is constituted by these two passages.

E's (90-92) recalls the objective of the transformation performed by An, recognising the effectiveness of her approach. To make all the other students focus on An's approach to identify it as an effective strategic model from which inspiration could be drawn, *T acts as a Reflective Guide*. She, in fact, asks An to share the reasons why she adopted this approach (93), fostering a further moment of argumentation at meta level, on the way of proving with algebraic language. An is able to reconstruct her proving process, explaining why she changed the representation, in reference to the final goal (*teleological r.)*. T reformulates An's explanation with the aim of fostering a real sharing between all the students (95).

(100) S: I did not understand.

(101) T: So … let's look at what An has done. We can try to repeat it. First of all she has considered the first case. If x is even, we can write it as 2n. So she has substituted 2n, obtaining 2n plus 2n squared. … Why did she take out this 2?

(102) St: So it is 2 multiplied by something …

(103) G: Because she wants to show that it is even.

(104) P: She could have taken out 2n (*instead of 2*)…

(105) T: Yes. But which was our objective? It was to show that the sum is …

(106) St: An even number!!!

(107) T: So we can take out what we need. If we take out 2, we can see that it is an even number. (*Then, they go on analysing the second part of the proof)*

When S declares that he did not understand, T choices again to act as a *Reflective Guide*, making the meaning of the expressions constructed by An explicit. To stress the reasons underlying the effectiveness of the transformations performed by An, T focuses again on the objectives of these transformations, playing the role of an *"Activator" of Anticipating Thoughts*. In this way, T is also acting as a *Guide in fostering a harmonized balance between the syntactical and the semantic level*: she both discusses the syntactical correctness of the performed transformations, referring to the two considered cases (*epistemic r.*) and the reasons why it is needed to distinguish between this two cases in relation to the final goal (*teleological r.*).

We stress that T's approach is particularly effective referring to the activation of the students' *teleological component* of rationality. St and G are, in fact, able to recognize the final goal of symbolic transformations (102-103-106).

CONCLUSIONS

We scrutinized the short episode in order to study: how the teacher deals with occasions of meta-level argumentations; what are the links between teacher's roles and students' rational behaviour. We may say that occasions for argumentation at meta-level arise when both students intervene and speak about their own or the classmates' proving processes, and when the teacher promotes them. When these occasions arise, the teacher adopts specific roles to foster meta reflection, so that students may become aware of their rational behaviour and share it with their mates.

In our analysis we also highlighted the links between the roles activated by the teacher and the different dimensions of rationality. When the teacher acts as an "*Activator of anticipating thoughts*", the *teleological component* of rationality is stimulated, since the goals of the syntactic transformations are shared and controlled.

When she acts as a *Reflective guide*, students from on one side share the reasons underlying the effectiveness of specific approaches (*teleological level*), on the other side better control the proving processes (*epistemic level*).

When she acts as a *Guide in fostering a harmonized balance between the syntactical and the semantic level*, she aims at making students develop new competencies in controlling the correctness of the activated processes (*epistemic level*) and in interpreting the meanings of the constructed expressions in relation to the problem situation (*epistemic and teleological level*). Also when she acts as an *"Activator of Interpretative Processes"*, she aims at making students activate the proper conceptual frames to correctly interpret the possible meaning of the constructed expressions (*epistemic level*).

In the future, we will go on with this work, with the aim of improving our analysis and of developing an in-depth reflection on the links we have highlighted between the teacher's roles and the students' rational behaviour.

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ENHANCING FORMATIVE ASSESSMENT STRATEGIES IN MATHEMATICS THROUGH CLASSROOM CONNECTED TECHNOLOGY

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The paper analyses how connected classroom technologies can be exploited to foster formative assessment practices in the mathematics classroom. Referring to a three-dimensional framework developed within the European project FaSMEd and to Hattie and Temperley's levels of feedback (2007), an excerpts from a classroom discussion in grade V is analysed in order to show the complex dynamics between the different formative assessment strategies that can be activated.

INTRODUCTION AND THEORETICAL FRAMEWORK

Research in Mathematics education has focused on the use of digital technology to support the Mathematics teaching-learning for many years. Within the European Project FaSMEd ("Improving progress for lower achievers through Formative Assessment in Science and Mathematics Education"), we investigate the use of connected classroom technologies (CCT) as means for supporting formative assessment (FA) practices in the mathematics classroom.

Within the FaSMEd Project, FA is conceived as a method of teaching where

> "[...] evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited" (Black & Wiliam, 2009, p. 7).

Following Wiliam and Thompson (2007), we adopt a model for FA in classroom context as consisting in *five key strategies*: (A) Clarifying and sharing learning intentions and criteria for success; (B) Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (C) Providing feedback that moves learners forward; (D) Activating students as instructional resources for one another; (E) Activating students as the owners of their own learning. This model identifies the three *main agents* (the teacher, the learners and their peers) and the three crucial processes in which the agents are involved: *Establishing where learners are in their learning*; *Establishing where learners are going*; *Establishing how to get there*.

On the other hand, some studies provide evidence about how new technology can be used as an effective tool in supporting FA processes (Quellmalz et al., 2012). Specifically, CCT may: create immersive learning environments that give powerful clues to what students are doing, thinking, and understanding, make students take a more active role in the discussions, encourage students, through immediate private

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 195–202. Szeged, Hungary: PME.

feedback, to reflect and monitor their own progress (Roschelle et al., 2004), and enable the teachers to monitor students' progress and provide appropriate remediation to address student needs (Irving, 2006).

Within the FaSMEd project, the Wiliam and Thompson's (2007) model has been extended in order to include the use of technology in FA processes. To this purpose, three different functionalities of technology have been identified: (1) *Sending and sharing*, when technology is used to support the communication among the agents of FA processes (for example: sending questions and answers, messages, files, displaying and sharing screens to the whole class or to specific students, sharing students' worksheets). (2) *Processing and analyzing*, when technology supports the processing and the analysis of the data collected during the lessons (such as the statistics of students' answers, the feedbacks given directly by the technology to the students, the tracking of students' learning paths). (3) *Providing an interactive environment*, when technology enables to create interactive environments in which students can work on a task and explore mathematical/scientific contents.

The result is a three-dimensional model taking into account three main dimensions (Fig.1): (1) *the five FA key-strategies* (Wiliam & Thompson, 2007); (2) *the three main agents* (the teacher, the student, the peers), and (3) *the functionalities* through which technology can support the three agents in developing the FA strategies.

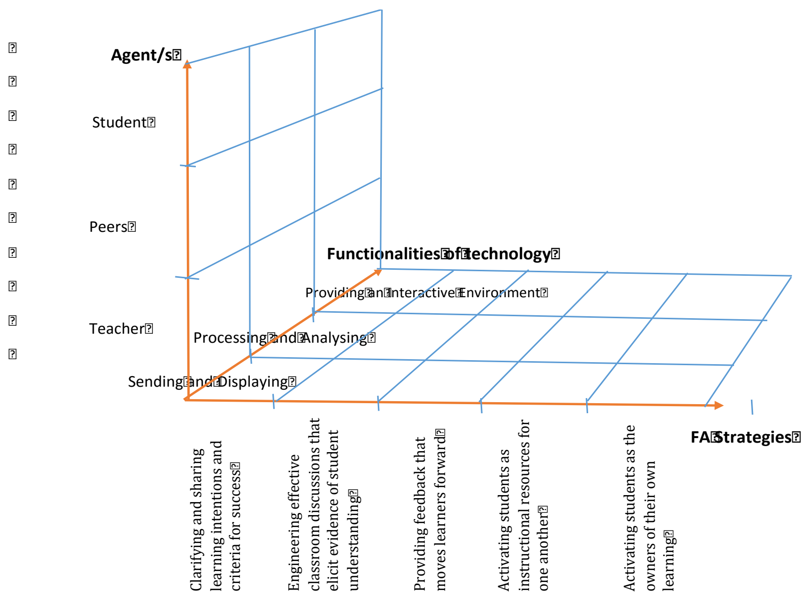


Fig. 1: Chart of the FaSMEd three-dimensional model

Feedback given by the different agents plays a crucial role for FA. Hattie and Temperley (2007) identify four *major levels of feedback*: (1) feedback *about the task*, which includes feedback about how well a task is being accomplished or performed; (2) feedback *about the processing of the task*, which concerns the processes underlying tasks or relating and extending tasks; (3) feedback *about self-regulation*, which addresses the way students monitor, direct, and regulate actions toward the learning goal; (4) feedback *about the self as a person*, which expresses positive (and sometimes negative) evaluations and affect about the student.

In this paper, we use the FaSMEd framework and the four levels of feedback in order to investigate the FA processes that take place in the mathematics classroom context, thanks to the support provided by CCT and to the teacher's choices. We also highlight the complex dynamical development between the different FA strategies activated by the agents involved.

METHODOLOGY

In Italy the FaSMEd project involves 18 teachers, from three different clusters of schools located in the North-West of Italy (from grade 4 to grade 7). Our hypothesis is that, in order to raise students' achievement, FA has to focus not only on basic competences, but also on metacognitive factors (Schoenfeld, 1992). Accordingly, we planned and developed class activities with the aim of: (a) fostering students' development of ongoing reflections on the teaching-learning processes; (b) focusing on making thinking visible (Collins, Brown & Newmann, 1989) and on students' sharing of the thinking processes with the teacher and the classmates. For these reasons, we explored the use of a CCT, which connects the students' tablets with the teachers' laptop and allows the students to share their productions, and the teacher to easily collect the students' opinions and reflections during or at the end of an activity. Each school was provided with tablets for the students and computers for the teachers, linked to IWB. In order to foster collaboration and sharing of ideas, students were asked to work in pairs or in small groups on the same tablet.

The use of the CCT was integrated within a set of activities on relationships and functions, and their different representations (symbolic, tabular, graphic), adapting activities from the ArAl project (Cusi, Malara & Navarra, 2011) and the Mathematics Assessment Program (http://map.mathshell.org). In line with our hypothesis and aims, our adaptation consisted in the creation of different worksheets belonging to three main categories: (1) *Worksheets focused on one or more questions* involving the interpretation or the construction of different representations of mathematical relationships between two variables (e.g. interpreting a time-distance graph); (2) *Helping worksheets*, aimed at supporting students, who meet difficulties with the type 1-worksheets, through specific suggestions (e.g. guiding questions); (3) *Worksheets prompting a poll* between proposed options.

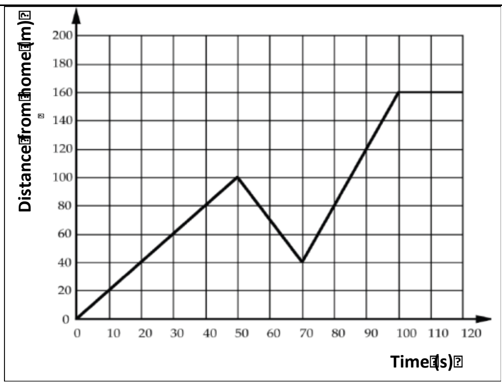
Usually the activity starts with a worksheet focused on one or more questions (type 1), sent from the teacher's laptop to the students' tablet. After facing the task and answering the questions, the pairs/groups send back to the teacher their written productions. The teacher can decide to send helping worksheets (type 2) to some groups, or the groups can ask for them. After all groups have sent back their answers, the teacher sets up a classroom discussion in which the students' written productions are shown and feedbacks are given. The discussion is engineered starting from the teacher's selection of some of the received written answers, to be shown on the IWB, and aims at highlighting: (a) typical mistakes; (b) effective ways of processing the tasks; (c) the comparison between the different ways of justifying claims. Polls are also used to prompt the discussion, in different parts of the lessons.

During the teaching experiments, one of the authors was always in the classes with the teachers, acting as a participant observer, namely taking notes, video-recording the lessons, and helping the teacher carry out the activities, for instance proposing interventions to foster fruitful discussions.

In this paper we will focus on a classroom discussion in grade 5. Specifically, in the next section we will analyse an episode that was selected because of the different FA strategies that are activated and the plurality of feedback that is provided. A variety of data were collected. We rely, in particular, on the qualitative analysis of the video-recordings, with the help of the written transcription of dialogues.

ANALYSIS OF AN EXCERPT FROM A CLASS DISCUSSION

The lesson is focused on time-distance graphs, introduced in the previous lesson through an experience with a motion sensor. The excerpt refers to the discussion of the first worksheet, reported in Table 1. As said, a researcher (first author of this paper) was present as a participant observer, and helped the teacher in managing the discussion.



Worksheet 1

Every morning Tom walks along a straight road from his home to a bus stop, a distance of 160 meters. The graph shows his journey on one particular day.

What happens in the period of time between 50s and 70s? How did you establish it?

Table 1: The worksheet sent to the students' tablet

From the mathematical point of view, the task and the discussion are aimed at: (1) Guiding the students in the interpretation of a time-distance graph; (2) Making the students focus on the processes underlying the correct interpretation of a time-distance graph, in particular with reference to the ascending/descending lines and to the information contained in the coordinates.

Four different answers are selected and projected on the IWB. The teacher asks the students to comment on them. One student, Livio, starts the discussion and proposes to focus on the following answer, which he declares (erroneously) *not* to be correct:

"Tommaso, in 20 seconds, was able to walk for 60 metres. We know that in 20 seconds he walked for 60 metres because we took 50s away from 70s, obtaining 20s, then we subtracted 60m from 100m and we obtained 40 metres".

Livio and his groupmate Giacomo declare that, in the period from 50s and 70s, Tommaso walked for 40m, not for 60m. Another student, Stefano, agrees with them, saying that the point (70,40) in the graph guarantees that Tommaso walked back for 40m. Almost all the students (even Vincenzo and Mirco, the authors of the answer) think that Livio and Stefano's arguments are correct. Only Arturo says that, in his opinion, the written answer is correct. In managing this part of the discussion, the teacher chooses to first invite those students who think that the answer is not correct express their point of view; then she asks Arturo to explain why, on the contrary, he thinks that the answer is correct.

145. Arturo: … if we look at the graph, he (*Tommaso*) arrives at 100m, then he goes back.

146. Teacher: Do we all agree that he goes back? (*A chorus of students answer "yes")*

147. Teacher: Who doesn't agree on the fact that he goes back? (*None of the pupils raises his/her hand*)

148. Arturo: However, he goes back to 40m, not for 40m (*stressing on the words 'in' and 'for'*). So we have to do the subtraction 100 minus 40. And the result is 60, not 40. So it (*the answer*) is correct.

149. Teacher: So is it (*the answer*) correct? Do you agree with Arturo? (*to the class*)

Silence.

150. Researcher: Please repeat the words you used (*speaking with Arturo*), since they are very precise. Listen to them (*speaking with the other students*).

Arturo repeats his reasoning, stating it slower and stressing the most important words, as asked. In particular, he explains that 60m is the result of the difference between 100m and 40m. At this point the projected answer is read again, and Vincenzo and Mirco (the authors of the answer) are addressed.

166. Researcher: You (*speaking to Vincenzo and Mirco*) said that you wanted to change your answer. Would you still change it or would you keep it as it is?

167. Mirco: We would keep our first answer.

168. Researcher: Ok. I have one question for all of you (*speaking to the whole class*): what is missing in this answer?

169. Mirco: That Tommaso went back! We did not write it.

170. Researcher: You did not say that Tommaso went back.

This part of the lesson exploits the *Sending and displaying* functionality of the technology: *sending* in a double direction, because the worksheets are sent to the students, who, in turn, send back their answers to the teacher's computer when they finish; *displaying* because the answers of the students are projected on the IWB and form the base for the class discussion. Projecting the collection of students' answers on the IWB enables the teacher, the researcher and the students to focus on different aspects, through the comparison of answers and justifications proposed by the students:

the *Sending and displaying* functionality appears to support the teacher (and the researcher) in activating the *FA strategy B* (*Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding*).

One student (Livio) erroneously identifies one correct answer as wrong, and explains what he identifies as a mistake. When many students (even Vincenzo and Mirco, who indeed gave the correct projected answer) agree with him, the teacher asks them to express their point of view: in this way, mistakes and misinterpretations come to the fore, and so she can gain information about *where the learners are in their learning*. Only afterwards, she exploits Arturo's disagreement to activate the *FA strategy D*: Arturo, in fact, is *activated as an instructional resource for his classmates* (lines 145, 148). His explanation (line 148), which highlights how to determine for how many meters Tommaso walked back, represents both a *feedback about the task* and a *feedback about the processing of the task*: *Strategy C* (*Providing feedback that moves learners forward*) is activated at the peers' level.

Seizing the effective and precise distinction made by Arturo in order to highlight that 40m, which is the distance from home, should not be confused with the walked distance, the Researcher (line 149) recognizes that the student has provided a correct argument, by asking him to repeat his words, and positively assessing them ("they are very precise"). In this way, she is activating *Strategy C*, giving students a *feedback about the processing of the task*, because she wants to make them focus on Arturo's way of interpreting the graph in order to understand what 40m represents. Afterwards, in order to activate *Strategy E* (*Activating students as the owners of their own learning*), the Researcher (line 166) asks Vincenzo and Mirco if they changed again their mind. By accepting Mirco's answer (line 167) without further questioning it or asking for additional justification (line 168), she is communicating, in an implicit way, that the answer is correct (*feedback on the task*). At the same time, she is prompting students to further focus on the same answer and look for something that is missing (again, *feedback on the task*). Mirco (line 169) shows that he really has *activated himself as the owner of his own learning* (*strategy E*) because he correctly identifies how his own answer can be completed.

In summary, the *Sending and Displaying* functionality of the technology supported the teacher in activating different FA strategies (in this case *strategies B*, *C*, *D* and *E*). We point out that the teacher is not the only agent involved and active during these processes. The students themselves, in fact, activate some strategies, because they give feedback to each other (*strategy C*, activated by peers), becoming instructional resources for one another (*strategy D*, again at the peers' level) and owners of their own learning (*strategy E*, activated by the students themselves). The following diagram (Fig. 2) illustrates the variety of these strategies, together with the agents and the functionality of the technology that is used.

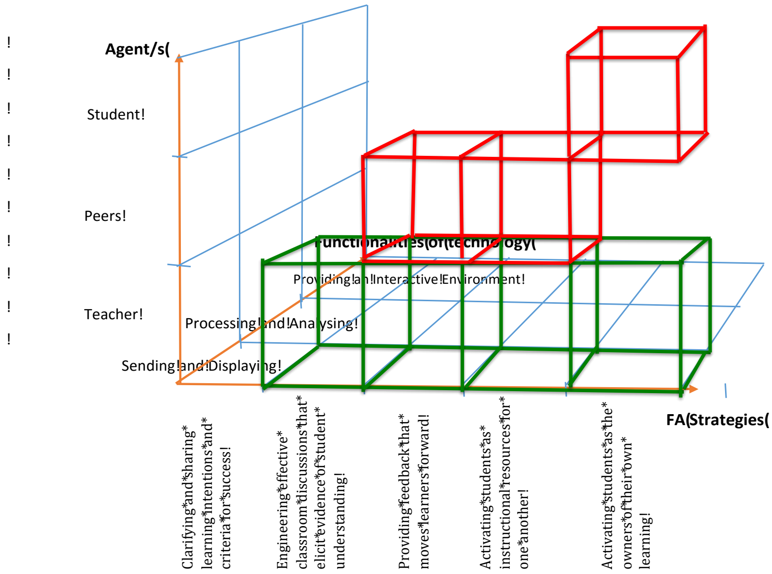


Figure 2: The FaSMEd three-dimensional model applied to the excerpt

CONCLUSIONS

The diagram (Fig.2) shows a global *static* picture of the episode, according to the FaSMEd framework. In our analysis we integrated this framework with the analysis of the levels of feedback, highlighting, in particular, feedback about the task and about the processing of the task. In other episodes from our teaching experiments we also gained evidence of feedback about self-regulation and about self as a person. As concerns the role of technology, the CCT we chose enables also to use the *processing and analysing* functionality through instant polls, which can be exploited by the teacher in engineering discussions rich in FA strategies (Aldon et al., in press).

The excerpt can also be analysed from a *dynamic* point of view. The following diagram illustrates the dynamic structure through which the class discussion has been engineered (*strategy B*) to activate other FA strategies:

The teacher asks to students to comment on a list of selected written productions, with the aim of activating the students as instructional resources for one another (*strategy D*).	→	The students provides feedback to each others and the teacher, too, comments, providing further feedback (*strategy C*).	→	The students, thanks also to the support provided by the teacher, exploit the provided feedback, activating themselves as owners of their learning (*strategy E*).

Table 2: The dynamic evolution of FA strategies in the analysed excerpt

In our view it is very important that strategy E is activated by the students themselves. From our results, it appears that working on strategies B, C and D (possibly A) is a promising road towards this goal. Further research is needed to confirm this hypothesis on firmer base.

Acknowledgment

Research funded by the European Community's Seventh Framework Programme fp7/2007-2013 under grant agreement No [612337]. We thank David Wright for having created a first digital version of Figure 1.

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STUDENTS' ABILITY TO CONNECT FUNCTION PROPERTIES TO FUNCTIONS IN DIFFERENT REPRESENTATIONAL MODES

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Recent research on the phenomenon of improper proportional reasoning focused on students' understanding of elementary functions and their external representations. So far, the role of basic function properties in students' concept images of functions remained unclear. We add to this research line by investigating how accurate students are in connecting functions to their corresponding properties and how this accuracy depends on function types and representations. Results show that students succeeded rather well in making the right connections between properties and functions. Errors depended on the type of function for which the properties were evaluated, but also on the kind of representation in which the function was presented. These results highlight the importance of function properties in students' concept images of functions.

THEORETICAL AND EMPIRICAL BACKGROUND

As a major mathematical model underlying various phenomena in real-life and in science and mathematics, proportionality justly receives a lot of attention in mathematics education worldwide. However, students' growing experience with proportionality during their school careers may have a serious drawback: It may lead to a tendency to use proportionality "anywhere", thus also in situations that are not proportional at all (Freudenthal, 1983). For example, many students believe that if the radius of a circle is doubled, its area is doubled too (De Bock, Van Dooren, Janssens, & Verschaffel, 2007) or that the probability to get at least one six in two dice rolls is two sixths (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). In the last decades, systematic empirical research has confirmed students' overuse of proportionality in a variety of mathematical subdomains, at distinct educational levels and in countries having different math educational traditions (for overviews and further analyses, see, e.g., De Bock et al., 2007; Van Dooren & Greer, 2010). Several explanations for students' overuse of proportionality were provided and discussed. De Bock et al. (2007) organized these explanations into three categories that referred to (1) the intuitive, heuristic nature of the proportional model for students, (2) students' experiences in the mathematics classroom and their beliefs toward mathematical modeling and problem solving, and (3) elements related to the mathematical particularities of the problem situation in which the proportional error occurs.

This last category is intrinsically mathematical and deserves our special attention in the context of this research report. Students have difficulties in understanding particular mathematical concepts, and when these concepts model situations that have something proportional in itself, such as the concepts of similar enlargement or

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 203–210. Szeged, Hungary: PME.

probability (see the two examples given above), the overuse of proportionality to non-proportional relations in the same context is easily made. This category of explanations is thus related to students' lack of a thorough understanding of basic mathematical functions or models that underlie a given situation.

Recently two studies focused on students' overuse on proportionality in the domain of functions, particularly on the mode in which a (non-proportional) function is represented (De Bock, Van Dooren, & Verschaffel, 2015). In a first study, students' ability to model textual descriptions of situations with different kinds of representations of proportional, inverse proportional, and affine functions. Results highlighted that students tend to confuse these models and that the representational mode has an impact on this confusion: Correct reasoning about a situation with one mathematical model can be facilitated in a particular representation, while the same representation is misleading for situations with another model. In a second study, students' ability to link representations of proportional, inverse proportional, and affine functions to other representations of the same functions was investigated. Results indicate that students make most errors for decreasing functions. The number and nature of the errors also strongly depend on the kind of representational connection to be made. Both studies provide evidence for the strong impact of representations in students' thinking about these different types of functions.

Unfortunately, the two studies by De Bock et al. (2015) gave us no insight into the criteria that students used while making connections between the various models and representations. Without being able to formulate clear hypotheses in that respect, it is likely that students did not depart from formal definitions of proportional, inverse proportional, and affine functions while making the required connections, but rather made use of their concept images of these function types. Tall and Vinner (1981) introduced the term *concept image* to describe "the total cognitive structure that is associated with a concept, which includes all the mental pictures and associated properties and processes. A concept image is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (p. 152). Typically, student's individual concept images are not globally coherent and have aspects that are not included in the formal concept definitions. In the case of proportional, inverse proportional and affine functions, students' personal concept images may include various properties they consciously or unconsciously associate with these functions and which they could deploy while making connections between these functions' representations, or, as formulated more generally by Adu-Gyamfi, Stiff, and Bossé (2012), "it is not the representations of a mathematical concept that are translated, but rather the ideas or constructs expressed in them" (p. 159).

In a recent study on processes and reasoning in representations of linear functions using task-based interviews, Adu-Gyamfi and Bossé (2014) found further evidence for the individual nature and lack of logical coherence in students' individual concept images. They revealed that students, enrolled in a pre-calculus course emphasizing and encouraging multiple representations, could make transitions from a given external representation of a linear function to another, but yet demonstrated little understanding of fundamental notions of (linear) functions. They concluded that, although all students in their study experienced the same classroom instruction, “some students demonstrated individual inconsistency within their own reasoning and used unconventional rationales in some scenarios and conventional rationale in others” (p. 189). In areas requiring representational versatility, i.e. the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations (Thomas, 2008), a new research perspective can shed a different light on students’ work and understanding. In that respect, research about how students are able to connect properties to functions given in different external representations could provide insights about the role of properties in students’ concept images of functions and about how students can employ these properties for solving related tasks.

In operational terms, this led us to the following research questions: (1) “How accurate are students in connecting functions to their corresponding properties?”, and (2) “Does this accuracy depend on function types and representations?” To answer these questions, we set up a new empirical study in which we investigated the role of the kind of function and the representational mode on students’ accuracy of connecting properties to functions.

METHOD

One hundred and eighty tenth graders (15- to 16-year olds) from four different schools in Flanders (Belgium) participated in the study. All participants followed general education with, depending on their study stream, four or five hours of mathematics per week. According to the Flemish educational standards these students had studied elementary mathematical functions including their applicability and their basic properties. They also gained experience with different representations of functions and learned to switch between representations.

Participants were randomly divided in three experimental groups and solved a written test. The test offered graphs, formulas or tables of four types of elementary functions (cf. infra), but every participant received these functions in only one of the three representations, depending on the experimental group he or she was assigned to. So each representation was included in one third of the tests. For each type of function participants had to evaluate the correctness of fifteen statements referring to function properties. These fifteen statements were classified into six clusters (see Table 1), but to the participants, they were presented in different random orders.

The same types of elementary functions as in the studies by De Bock et al. (2015) were included in this study: proportional functions (“$y = ax$” with $a > 0$) and three types of non-proportional functions, namely inverse proportional functions (“$y = a/x$” with $a > 0$), affine functions with positive slope (“$y = ax + b$” with $a > 0$ and $b \neq 0$), and affine functions with negative slope (“$y = -ax + b$” with $a > 0$ and $b \neq 0$). The non-proportional functions share certain characteristics with proportional functions but not all. Affine functions with positive slope for instance share with proportional functions the property that their graph has the shape of a straight line, and that the same increase Δx in x always results in the same increase Δy in y. But while in proportional functions, doubling x implies doubling y, this does not hold for affine functions with positive (or negative) slope. Still, students may nevertheless assume this. To allow a proper interpretation of possible confusions between function properties, the parameters a and b of the representatives of the four function types in the test were held constant, being respectively 2 and 3 (see Table 1).

Responses were statistically analysed by means of analyses of variance, using the generalized estimating of equations (GEE) approach within SPSS (Liang & Zeger, 1996), with "Type of function" as an independent within subject variable, "Representation" as an independent between subject variable and students' "Accuracy score" as the dependent variable. These analyses were performed on the total scores, the cluster scores and the scores on the individual statements. Significant main and interaction effects were further analyzed by means of pairwise comparisons.

Qualitative statements

- (*QU1*) If x increases, then y increases too.
- (*QU2*) If x increases, then y decreases.

Additively increasing statements

- (*AI1*) If x increases with one unit, then y increases with one unit too.
- (*AI2*) If x increases with one unit, then y increases with two units.
- (*AI3*) If x increases with one unit, then y increases with three units.

Additively decreasing statements

- (*AD1*) If x increases with one unit, then y decreases with one unit.
- (*AD2*) If x increases with one unit, then y decreases with two units.
- (*AD3*) If x increases with one unit, then y decreases with three units.

Multiplicatively increasing statements

- (*MI1*) If x becomes two times larger, then y becomes two times larger too.
- (*MI2*) If x becomes three times larger, then y becomes three times larger too.

Multiplicatively decreasing statements

- (*MD1*) If x becomes two times larger, then y becomes two times smaller.
- (*MD2*) If x becomes three times larger, then y becomes three times smaller.

Statements referring to the intercept

- (*IC1*) If x equals 0, then y equals 0 too.
- (*IC2*) If x equals 0, then y equals 2.
- (*IC3*) If x equals 0, then y equals 3.

Table 1: Overview of the fifteen statements in their respective clusters

Note. Codes between brackets were added to facilitate references to these statements in the next section.

RESULTS

Table 2 gives an overview of students' total scores (in %) for the four types of functions (proportional, inverse proportional, affine with positive slope, and affine with negative slope) and the three types of representations (graph, formula, and table). These scores reveal that students performed rather well in making the right connections between properties and functions: The total percentage of correct assignments in the whole sample amounted to 91.5% and none of the percentages was less than 85%. But still, percentages varied significantly across function types and representational modes. The GGE analysis on the total scores indeed revealed a main effect of "Type of function", Wald $X^2(3, 180) = 103.163$, $p = .000$: For the inverse proportional function the accuracy rate was lower than for the other three function types ($p = .000$) for which accuracy rates didn't differ significantly. This analysis also revealed a main effect of "Representation", Wald $X^2(2, 180) = 47.938$, $p = .000$. This means that the accuracy rate of connecting properties to functions depended on the representational mode in which the function was given. The highest accuracy rate was observed for functions that were given in a tabular representation, which significantly differed from those given in a formula ($p = .003$) or graphical ($p = .000$) representation. The difference between these latter two was not significant. This result is in line with that of De Bock et al. (2015) and is likely due to the fact that tables provide a set of concrete function values which facilitate checking the correctness of function properties. Finally, a significant interaction between "Type of function" and "Representation" was observed, Wald $X^2(6, 180) = 17.952$, $p = .006$: The effect of "Function type" was thus not the same in all representations (and vice versa). For the proportional and affine function with positive slope, both the tabular and formula representation elicited significantly more correct assignments than the graphical representation ($p = .002$ and $p = .037$), while for the affine function with negative slope, only the tabular representation elicited significantly more correct assignments ($p = .017$). For the inverse proportional function, no significant differences between representations were found.

	Graph	Formula	Table	**Average**
Proportional	90.3	95.1	95.7	**93.7**
Inverse proportional	85.0	85.8	90.6	**87.1**
Affine (positive slope)	88.0	94.6	97.3	**93.3**
Affine (negative slope)	89.2	89.8	96.1	**91.7**
Average	**88.1**	**91.3**	**94.9**	**91.5**

Table 2: Overview of students' total scores (in %)

Table 3 gives an overview of students' scores (in %) for each statement and for each cluster of statements. Percentages of correct assignments for each cluster are high, but compared to Table 2, a lower minimum percentage (80.5%) and a higher maximum percentage (97.0%) – and thus also a higher range (16.5%) – is revealed. Qualitative statements as well as statements referring to the intercept elicited most correct assignments, while both clusters of multiplicative statements elicited least correct assignments. In line with our study's focus, we subsequently analyzed for each cluster the role of the kind of function and the representational mode on students' accuracy to connect properties to functions. These analyses of variance showed that the effects that were observed in students' total scores (main effects of "Type of function" and of "Representation", as well as the interaction effect between these two variables), were also present in each of the six clusters separately. Because of space limitations, we will not elaborate on corresponding statistics and results of pairwise comparisons for all these main and interaction effects. Instead, we will limit ourselves to discussing the most notable results. *First*, in contrast with the results in the whole sample, the two increasing clusters elicited most correct answers for the two *decreasing* functions. So, students rather easily discovered that the increasing statements were not applicable to the decreasing functions, likely on the basis of the information about the general function behavior included in these statements. *Second*, the proportional function elicited fewest correct assignments in the multiplicative increasing cluster. Relatively many students were thus not able to correctly assign the "proportional properties", as expressed in the multiplicative increasing statements, to proportional functions. This result is striking because these properties are most prototypical for proportional functions and also because students often over-rely on these functions and their properties. *Third,* it is notable that the cluster-specific results with respect to the representational modes were broadly the same as the ones found in the sample as a whole. More specifically, the tabular representation elicited most correct assignments in all clusters.

Qualitative statements	*QU1*	95.6	**96.0**
	QU2	96.3	
Additively increasing statements	*AI1*	91.4	**92.2**
	AI2	87.4	
	AI3	97.5	
Additively decreasing statements	*AD1*	92.4	**93.8**
	AD2	91.4	
	AD3	97.1	
Multiplicatively increasing statements	*MI1*	85.3	**84.9**
	MI2	84.3	
Multiplicatively decreasing statements	*MD1*	82.8	**80.5**
	MD2	77.9	
Statements referring to the intercept	*IC1*	94.6	**97.0**
	IC2	98.8	
	IC3	97.2	

Table 3: Overview of students' scores (in %) for each statement and for each cluster

To deepen out the cluster-specific results, additional analyses were conducted on the level of the individual statements. As shown in Table 3: Accuracy rates clearly differed between statements. Statement MD2 elicited fewest and statement IC2 elicited most correct assignments (respectively 77.9% and 98.8%), so the range amounted to 20.9%. A discussion of each statement separately is not feasible, so we limit ourselves to statement MD2 that elicited a minimal accuracy rate (a statistical analysis on the scores of statement IC2 that were extreme in the opposite direction, was not possible because of ceiling effects). The analysis of variance on the scores of statement MD2 only showed a main effect of “Type of function”, Wald $X^2(1, 180) = 40.978$, $p = .000$. The inverse proportional function, for which this statement is correct, elicited the highest percentage of incorrect assignments. Clearly, students struggle with this formulation of the “inverse proportional property” (referring to tripling, while this type of function was represented in the test by $y = 2/x$).

CONCLUSIONS AND DISCUSSION

This study complements the studies by De Bock et al. (2015) by pointing to the mediating role of representations in students’ understanding of properties of proportional, inverse proportional and affine functions, but results also provoke new questions subject for further research.

In response to the first research question, we observed that students perform rather well in making the right connections between properties and functions. This result suggests that the basic function properties are part of most students’ concept images of the above-mentioned functions. However, it helps little to explain why, in De Bock et al.’s (2015) studies, students often encountered difficulties when being asked to connect models and representations, on the contrary, it raises questions about the hypothesis that students employed function properties for making this type of connections. A plausible reason for the positive result in this study, compared with the results of previous studies, is that function properties were addressed *explicitly*, while in most of the previous studies as well as in current educational practices, these properties are often used in an implicit or even unconscious way.

In response to the second research question, function types and representations affected accuracy rates, and in that sense, results of this study are largely in line with the ones reported in De Bock et al.’s (2015) studies. With respect to function types, the inverse proportional function proved to be most problematic for students. This result is likely due to the intrinsic difficulty level of this type of function, but might also be influenced by educational practices too in the sense that Flemish students study functions of the form $y = a/x$ in Grade 10, but a systematic study of rational functions is 11^{th}-grade subject matter. However, not only function types, but also representational modes, as a part of students’ concept image of these functions, affected accuracy rates.

Although this study provides some clear answers to the two research questions, it also has its limitations related to its nature and design. A first limitation relates to the absence of qualitative data that could have shed a light on the kind of strategies that students employed to make the right connections between representations of functions and corresponding properties, strategies most of these students were able to apply in the context of this study, but apparently not or not to the same extent in previous research in which these strategies could have been helpful too. A second limitation relates to the generalizability of the results that is negatively affected by the artificial nature of the task that students had to fulfil, a kind of task that clearly differs from genuine classroom tasks in which functions and their properties are involved.

An implication that could be drawn for mathematics education practice is the need for drawing sufficient instructional attention to properties of functions and to explicitly discuss differences between properties of proportional and various types of non-proportional functions. Exploring function properties in different representational modes can point to similarities, but also to crucial differences and thus strengthen students' insights in these properties and their appearances.

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“*THINK ABOUT YOUR MATH TEACHERS*”

A NARRATIVE BRIDGE BETWEEN FUTURE PRIMARY TEACHERS’ IDENTITY AND THEIR SCHOOL EXPERIENCE

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Pre-service teachers approach their professional learning in mathematics with a complex set of needs and wants. These needs and wants are strongly affected by the tension deriving from the realisation of the gap between what an individual wants to become as a mathematics teacher (his/her ideal of mathematics teacher) and what he/she believes to be at present. Professional identity as a mathematics teacher can be seen as a continuous development arising from this gap. For these reason, both as researchers and as teacher educators, it appears significant to study what ideals of positive and negative mathematics teachers the future teachers have.

INTRODUCTION AND THEORETICAL FRAMEWORK

Pre-service teachers approach their professional learning in mathematics with a complex set of needs and wants. In particular, they approach their path trying to satisfy their own wants and needs in the light of their previous experiences as math students (Liljedahl, 2014).

These previous experiences largely determine future teachers’ mathematical identity (Kaasila, 2007). According to Kaasila, we define mathematical identity as the set of narratives that pre-service teachers create to describe themselves as mathematics learners and teachers. In particular, identity as a mathematics teacher can be seen as a continuous development arising from the gap between the ideal of good mathematics teacher that a pre-service teacher has into his/her mind and the teacher the individual thinks to be at the present moment of his/her formation (Sfard & Prusak, 2005).

In this framework the case of future primary teachers appears particularly interesting: several researches highlight that many of them lived hard experiences with mathematics, developing negative emotions towards mathematics and towards the fact that they will have to teach mathematics (Coppola et al., 2013). Therefore, there is often a strong tension between what the individual is and what he/she wants to become as a mathematics teacher (Krzywacki & Hannula, 2010). As underlined by Liljedahl et al. (2014), the management of tensions defines pre and in-service teachers’ wants and needs and affect their decisions (respectively in their approach to professional learning and in their school practice).

The study of future primary teachers’ mathematical identity as mathematics teachers appears particularly significant to understand how they approach to opportunities offered them during the professional development (Lutovac & Kaasila, 2014). As Krzywacki and Hannula (ibidem) underline, pre-service teachers’ identity as

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 211–218. Szeged, Hungary: PME.

mathematics teachers is strongly influenced by the real mathematics teachers met during the school period.

For these reasons, as researchers in mathematics education, we believe that it is interesting to compare future primary teachers' viewpoints about what school teachers need to teach mathematics effectively with the existing great amount of literature on that issue (Ball & Bass, 2000; Mason, 2008; Oliveira & Hannula, 2008). On the other hand, as teacher educators, it is crucial to offer to future teachers the opportunities for reflection on: their own learning, their experiences with understanding of mathematics, as well as on the approaches used by their teachers to introduce and discuss topics (William, 2001).

Therefore, within a wider study about teachers' mathematical identity, beliefs and emotions towards mathematics, we have developed a narrative study aimed at identifying which traits future primary teachers consider distinctive of effective mathematics teachers and which traits they consider distinctive of ineffective mathematics teachers.

METHODOLOGY

Procedure and population. The study developed through two different phases. In this paper we will focus on the second one, but we believe that, in this section, it is important to briefly sketch the study in its wholeness.

The first phase involved 212 future primary teachers enrolled in the first year of the university degree for primary school teachers of six different Italian universities. They were asked to answer in anonymous way to a questionnaire composed by 7 open questions about their past experiences, beliefs and emotion towards mathematics, within 1 hour. In particular, Q2 was: "*What has been your past experience with mathematics during the school period? Can you describe an episode occurred during your school period that you consider crucial in the development of your current relationship with mathematics?*". By the analysis of the collected data, it emerges as 'mathematics teacher' is the most recurrent factor: we found 131 occurrences of teachers on 212 answers to Q2 (the 62% of the collected narrations). There are other recurrent aspects in the episodes narrated (such as successes or failures in math, specific topics, the transition from a school level to another one), but mathematics teacher results by far the most recurrent factor both in *negative* episodes and in *positive* one. This result affected the development of the second phase of our study, that it is the focus of this paper. We decided to investigate about which traits future primary teachers consider distinctive of effective (ineffective) mathematics teachers stimulating a reflection about the approaches used by their mathematics teachers, possibly recognizing both *positive* and *negative* model of mathematics teacher in their school experiences.

The second phase involved 59 future teachers enrolled in the first year of the university degree for primary school teachers of seven different Italian universities: 44 of them (75%) answered to an online questionnaire composed by 11 questions (9 open-ended and 2 close-ended questions) and 15 of them (25%) answered to a semi-structured interview concerning the same topics of the online questionnaire. The interviews had not a settled time: it varied in a range from 25 to 65 minutes. The interviews were audio-recorded and then fully transcribed.

In this paper, we focus on the future teachers' answers to the questions Q6: "*Think about your math teachers. Is there one you would like to become like? In what way? Why?*" and Q7: "*Think about your math teachers. Have you ever thought 'I never should act with my students like s/he did with me'? Why?*". Questions Q6 and Q7 were included both in the questionnaire and in the oral interviews.

Rationale. The choice of the research instruments is never neutral. In our case, the narrative approach is not only coherent with the definition of mathematical identity assumed: as Kaasila (2007) underlines, through this methodological approach, what pre-service teachers consider really important in their experiences comes to the fore. Individuals develop their sense of identity by describing themselves as protagonists of different stories: what creates the identity of the individual is the identity of the story, not the other way around. We chose to use both an open-ended questionnaire and interviews because we believe that the two instruments complement each other. As a matter of fact, the use of questionnaire permits to collect a wider range of answers and, according to Cohen et al. (2007), an open-ended question can catch the authenticity, richness, depth of response, honesty and candor which are the hallmarks of qualitative data. On the other hand, questionnaires have their limitations: they are still one-way, when compared with interviews. Moreover, Kaasila (ibidem) has highlighted the potential of narrative interviews for the study of pre-service teachers' mathematical identity.

Regarding the analysis of the narrative data collected, we refer to the work of Lieblich et al. (1998). They recognize two main choices related to two independent dichotomies. The first choice concerns the narrative unit of analysis: holistic (the narrative is analized as a whole) vs categorical (specific utterances are singled out from the complete narrative) analysis. The second choice concerns the traditional dichotomy between the attention to the content or the attention to the form of a narrative. Our analysis approach was mainly content-categorical oriented, being considered particularly suitable to study a phenomenon common to a group of people (Kaasila, ibidem).

RESULTS AND DISCUSSION

Not all respondents answer affirmatively to Q6 or Q7: some of them explicit that they did not recognize a *positive* or *negative* model of mathematics teacher in their school memories (FTQ19[4]: "*Actually I have never met great mathematics teachers: for one reason or the other, they have never fully satisfied me*"). Within our sample, it emerges

[4] Here, as well as in the next excerpts, the letter Q refers to Questionnaire, the letter I refers to Interview and the number indicates the progressive numbering of the respondents.

a gap between the percentage of respondents that does not recognize positive model in their mathematics teachers (32%) and the percentage (only the 12,5%) of those who declared they have not memories of negative model of mathematics teachers (in these percentages, we had not considered positive or negative references to academics). In the light of these data, it seems that, reflecting on their experiences, future primary teachers have greater ease in recognizing the negative traits in the teaching styles their mathematics teachers used, rather than the positive ones.

The analysis of the data collected through the questionnaire and interviews permits to describe a long list of traits that future teachers associate to their mathematics teachers (see table 1 below). The answers to Q6 and to Q7 permit to identify the traits associated respectively to positive models of mathematics teacher (positive traits) and to negative models of mathematics teachers (negative traits).

Positive traits	**Negative traits**
Competence in math	Incompetence in math
Competence in teaching math	Incompetence in teaching math
Clarity in explanation	Ambiguity in explanation
Interactive teaching methods	Frontal teaching method
Ability to show the link between math and real life	Inability or disregard in going beyond the content included in the syllabus
Relational Approach	Instrumental Approach
Passion for math	Coolness for math
Passion for teaching math	Coolness for teaching math
Serenity	Aggression
Severity	Severity
Attention to students' needs and difficulties	Indifference for students' needs and difficulties
Confidence in students' capability	Doubts about students' capability
Ability to develop a good relationship with students	Inability to develop a good relationship with students

Table 1: Duality between positive and negative teachers' traits.

It clearly emerges a duality between positive and negative traits. There is a unique *anomaly*: severity. Some respondents consider severity as a negative trait that can contribute to create a bad climate in the classroom (FTQ9: "*I don't want to be like my primary teacher: she is rude and severe. I was intimidated by her, therefore I was stuck, I went into a panic*"). Other respondents whereas underline their conviction that a certain level of severity is needed to be a respected and effective teacher (FTQ15: "*She was an excellent teacher: she was severe and very good in teaching*").

Analysing traits in table 1, we can recognize some aspects included in the Mathematical Knowledge for Teaching model (Ball & Bass, 2000). We observe that the references to the common content knowledge are mainly stressed in the answers to Q7 rather than in the answers to Q6, and they are almost always combined to pedagogical aspects (FTQ37: "*My secondary teacher was incompetent and unable to interact with teenagers*"). Future primary teachers seem to be aware that having a solid content knowledge is a necessary but not a sufficient condition to be an effective teacher, in particular at primary school level. This is also evidenced by the greater number of recurrent traits related to pedagogical content knowledge or to affective aspects.

A significant outcome of our survey is the attention given by future teachers to the view of mathematics their teachers offered. In his famous paper, Skemp (1976, p.6) stated: "*I now believe that there are two effectively different subjects being taught under the same name, 'mathematics'*", introducing the concepts of relational and instrumental mathematics. According to this classification, within our sample, we found a general appreciation for mathematics teachers that have proposed a relational approach to mathematics (FTQ42: *"She always got in-depth when explaining. The first question was always "Why?" and never "How we have to solve it?"*, FTQ22: "*I have appreciated my mathematics teacher from the beginning because she tried to teach us to look beyond memorization of formulas*"), and conversely a widespread criticism toward teachers with an instrumental approach to math (FTQ27: "*I have had teachers that forced me to memorize formulas and to recite rules*").

On the other hand, discussing why an instrumental approach to mathematics appears to be often so appealing for teachers and students, Skemp describes some apparent *advantages* of this approach to mathematics. In particular he underlines that within its own context, instrumental mathematics is usually easier to understand and the rewards are more immediate. The analysis of the interviews shows as the appreciation for "relational mathematics teachers" is often the result of a posteriori reflections, based on a greater awareness (FTI7: "*For a long time I thought that mathematics was characterized by memorization. Probably I was focused on memorization rather than understanding (…) Now I'm understand that I have never found teachers that try to explain me the reasons beyond mathematical facts (…) many facts were simply assumed (…) but I was not able to understand: perhaps I needed further and different explanations*"). Sometimes the awareness of the weakness of an instrumental approach to mathematics emerges from an a posteriori comparison with the educational results of a relational approach (FTI1: "*My mathematical experience has been linear and planned (…) I was very relaxed and satisfied. My sister had a more troubled path, but this path permits to her and many of her classmates to develop the flexibility that I haven't (…) Probably I have a more complete preparation in mathematics (…) but, when I have to manage with a new situation, a problem that goes beyond the* *application of memorized schemas, I am disoriented, whereas she is more ready, prepared and reactive because of her flexibility*”).

As we anticipated, a great emphasis to affective aspects emerges in future teachers’ answers. In particular, *passion* and *calm* are considered crucial quality for an effective mathematics teacher. Passion for mathematics and passion for teaching mathematics are considered both essential to convey passion for mathematics to the students. However, these two passions are not always coincident (FTQ4’s answer to Q7: “*my teacher surely loved mathematics, but he was not interested in its teaching at all*”). A calm teacher's presence is considered a key element in order to have an appropriate classroom climate and to promote passion for mathematics (FTQ26: “*She had charisma and she was so calm that I was enchanted during her math explanation*”). Particularly interesting that, in the answers to Q7, all the future teachers’ memories involving an aggressive teacher are related to experiences at the primary level (FT15: “*I've always promised myself that I won't behave like my primary teacher. I would avoid to result aggressive and intimidate pupils*”).

On the other hand, the most stressed traits in the affective side concern two aspects of the teacher’s *attention* to the students.

The first one is the teacher’s confidence in students’ mathematical potential (FTQ37: “my teacher have always really appreciated my math ability. Therefore I would like to have this talent in showing the appreciation for student’s ability”). In particular, the richness of the data collected through interviews permitted to highlight the strong emotions elicited in students when they reach the awareness that teacher (and adults in general) has low confidence in their mathematical abilities. As underlined by FTI6, this awareness can persuade the student to be not able in math: “I was dealing with people who had no confidence in me. This fact demoralized me a lot. Judgments like ‘Okay, after all she is not able to understand’ or ‘Okay, after all she is not able to do the appropriate reasoning’ convinced me that I was not able to do math (…) When I met someone that believes in me, I gained confidence in my abilities (…) I believe that it is important to interact with a teacher that believes in you, in particular this support is fundamental at the primary school”.

The second one is teacher’s attention to students’ mathematical difficulties (FTI31: “She had a positive attitude towards students’ difficulties, she was inclusive: when we had difficulties in assimilating some topics, she tried new teaching methods”; FTQ17: “She wasn't interested in the development of our cognitive processes but she only focused in finishing the curriculum. She just wrote on the blackboard and she did not consider our difficulties. When we asked her something, she always answered that she hadn't got time to reply”).

To conclude, we want to underline that in the interviews future teachers find the time to describe the evolution of their convictions about positive and negative ways of teaching mathematics. Particularly interesting in this sense is the episode narrated by FTI13: “*It happened that during my practicum I was paired with the primary teacher* *that I have had when I was student. At that time, I liked very much her teaching methods and I liked the activities that she proposed to us. But in my practicum I have seen…I had a flashback: she proposed the same experiences that she used with us 15 years earlier. Identical! Identical! I have thought: 'No, I don't want to do this…ever!'*"

CONCLUSION

There are a lot of papers in the field of mathematics education focused on what characteristics primary teachers should have in order to teach mathematics effectively. These studies have the ambition to affect the way teachers' education programs are developed. We strongly believe that it is highly relevant to listen the voice of the future teachers about this issue. "Teachers do not approach their professional learning as blank slates" (Liljedahl et al., 2015, p. 193): their beliefs and opinions about their experiences affect their wants and needs in the professional development setting. Knowing these wants and needs and their relationship with future primary teachers' experiences with mathematics is significant both as researchers and math educators.

In our study, the request to produce a mathematical autobiography forced the respondents to re-enact and re-consider their own past experiences, in order to develop a new awareness about their own wants and needs. In particular, it emerges that future primary teachers go beyond the boundary delimited by mathematical knowledge for teaching in their reflections, placing a strong emphasis to affective aspects in their judgments about mathematics teachers.

The goal is not to draw a more complete list of what primary teachers need to teach mathematics effectively. According to Mason (2008, p. 317), future teachers' attention needs to be focused not on a list of prescriptions, but on noticing: "The aim of teacher education is to prepare the ground so that novice teachers will find themselves increasingly sensitised to noticing possibilities for initiating, sustaining or completing actions which they might not previously have had come to mind". On the other hand, as Liljedahl (2014) underlines, the recognition of future teachers' wants and needs should have an impact on how we view our role as facilitators.

In particular, it appears fundamental to create bridges between the research results and the action for improving practices. Our study underlines the need to incorporate in a systematic way *affect* in the education program for future primary teachers. Some steps in this direction are being taken (Gómez-Chacón, 2008), but we believe that many more will need to be taken soon.

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CONCEPTUALIZING PROFESSIONAL CONTENT KNOWLEDGE OF SECONDARY TEACHERS TAKING INTO ACCOUNT THE GAP BETWEEN ACADEMIC AND SCHOOL MATHEMATICS

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Answering the question as to what content knowledge (CK) mathematics teachers need is essential for designing teacher education programs as well as for investigating teachers' professional knowledge. Hence, conceptualizing professional CK is a central topic in mathematics education. Existing conceptualizations of secondary teachers' mathematical CK diverge, however, widely – in particular regarding the extent to which school mathematics or academic mathematics is a reference point. This theoretical paper traces different ideas to bridge the gap between academic and school mathematics and reviews prominent conceptualizations through this lens. The emerging need for considering a CK linking academic and school mathematics is addressed by the introduction of a construct called school-related content knowledge.

INTRODUCTION

Most studies of teachers' professional knowledge use models that draw on the categories identified by Shulman (1986) (Depaepe, Verschaffel, & Kelchtermans, 2013). In the case of mathematics education in particular the discipline-specific constructs content knowledge (CK) and pedagogical content knowledge (PCK) are in the focus. However, the question as to what discipline-specific professional knowledge teachers need has been addressed by scholars in mathematics education already before Shulman's (1986) seminal work (e.g., the conference proceedings Bauersfeld, Otte, & Steiner, 1975). Fletcher (1975), for instance, argued that mathematics teachers need a special kind of mathematical knowledge:

> The mathematics teacher requires a general knowledge of mathematics in order to be able to communicate with other mathematicians and also to establish his credentials; but he also requires special knowledge of certain areas of mathematics, in the way that an engineer or an astronomer requires special knowledge. […] It is part of our problem that the teacher's special mathematical knowledge is inadequately defined and insufficiently esteemed (p. 206).

About thirty years later Ball and her colleagues from the University of Michigan (e.g., Ball, Thames, & Phelps, 2008) did a job analysis of elementary mathematics teachers in the USA and identified the so-called *specialized content knowledge* (SCK) that these teachers need. This research, however, does not cover the conceptualization of secondary teachers' CK, as in this case typically academic mathematics serves as a more important reference. Consequently, this article resumes currently unconnected ideas regarding a special kind of CK that secondary mathematics teachers need and

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introduces a corresponding construct called *school-related content knowledge*, which may be particularly interesting with respect to pre-service teacher education.

THE GAP BETWEEN ACADEMIC AND SCHOOL MATHEMATICS

It is well-known that mathematics as a scientific discipline is usually very different from mathematics as a school subject (e.g., Bromme, 1994; Schweiger, 2006; Wu, 2011). Although it may depend on culture-specific traditions of secondary mathematics how big the gap between academic and school mathematics is, they typically differ not only with respect to the contents and their degree of abstraction, but also regarding their epistemology (e.g., Bass, 2005; Bromme 1994; Wu, 2011): Mathematics as a scientific discipline is characterized by its axiomatic-deductive structure and generally marked by a high level of abstraction. Mathematics as a school subject usually puts its main focus on applying mathematics as a tool for understanding and facilitating everyday live. Hence, mathematical concepts are often introduced inductively by means of prototypes and bound to a certain context.

In view of this gap, the question arises as to what kind of mathematics prospective secondary mathematics teachers should be taught. Is it school mathematics? Or academic mathematics? Or both? Or something else? This kind of questions were already asked by Otte (1979): “How in particular is his [the mathematics teacher’s] knowledge related to the content of the mathematics school curriculum and to mathematics as a science?” (p. 119). Considering the subject-specific parts of secondary teacher education in different countries shows a large variation in the extent to which school mathematics or academic mathematics is focused (Blömeke, Hsieh, Kaiser, & Schmidt, 2014). However, there is a broad consensus among researchers in mathematics education concerning the following: (i) school mathematical knowledge is not sufficient as CK of secondary mathematics teachers (e.g., Dörfler & McLone, 1986; Shulman, 1986), (ii) teaching pre-service teachers academic mathematics does not necessarily enable them to make connections with the school mathematics they are supposed to teach (e.g., Klein, 1908/1932; Wu, 2011). Hence, both, school mathematics and academic mathematics play an important role for secondary teacher education, but it is not enough to treat them separately. Thus it was and still is a central question as to how the gap between these two kinds of mathematics can be bridged.

Discussions about what CK teachers need emerged for instance in the context of the development of new curricula in the 1960s and 1970s. It was emphasized that teachers should not merely know school mathematics, but also about its structure, i.e. the curricular order of contents and their interdependencies (e.g., Fletcher, 1975). In order to understand this structure, there is, however, also knowledge needed about reasons for this curricular structure, which are rooted in fundamental ideas of academic mathematics (Bruner, 1960; Schweiger, 2006). Focusing on such fundamental ideas of mathematics (e.g., measuring, function) also facilitates bridging the gap between academic and school mathematics (Schweiger, 2006). However, from the perspective of academic mathematics, the structure of school mathematics remains inconsistent (e.g., Wu, 2011). Thus, even though the perceived gap between academic and school mathematics can be reduced by knowing about the structure of school mathematics and its legitimation rooted in fundamental ideas of academic mathematics, it cannot be closed entirely this way (Schweiger, 2006). In order to understand and deal with the inconsistencies between academic and school mathematics, teachers should be able to bridge the gap locally, i.e. on the level of specific contents. This can be done by making connections in top-down and in bottom-up direction (i.e., with academic vs. school mathematics as a starting point). Taking academic mathematics as a starting point, the question as to how such mathematical contents can be transformed for the mathematics classroom has always been central for scholars of mathematics education (e.g., Dörfler & McLone, 1986; Fletcher, 1975; Freudenthal, 1973). In view of these considerations, teachers should know how to reduce certain contents of academic mathematics for teaching purposes and in particular about consequences of such reduction. Since teachers often have to deal with already reduced mathematical contents they encounter in textbooks and learning environments, they also need to examine whether these contents have been transformed in a mathematically appropriate way. In this case school mathematics can be seen as a starting point and connections with academic mathematics must be made in bottom-up direction. Hence, teachers also have to know which mathematical definitions, theorems, and proofs lie behind the contents in school mathematics. Such bottom-up connections were considered for instance already in the 1970s, when the concept of 'mathematical background theories' (German: 'Hintergrundtheorien') of school content was introduced (e.g., Vollrath, 1979). According to Vollrath (1979) this notion (in a broad sense) was used to describe the complex of mathematical concepts, statements, interrelations, methods, and representations that lies behind a mathematics teaching sequence in class (p. 8 – 9).

CONCEPTUALIZATIONS OF TEACHERS' PROFESSIONAL CK

In the past decades, many scholars and researchers suggested conceptualizations of the mathematical CK required for teaching in the mathematics classroom. As mentioned above, Shulman's (1986) structuring of teachers' content-specific professional knowledge lies at the heart of many models of teacher professional knowledge. Although his work is mainly appreciated for the introduction of PCK, he also emphasized the significance of CK. His conceptualization of teachers' CK focuses on academic knowledge. Seeing the structure of the school subject as being derived directly from the structure of the academic discipline (Shulman, 1986), there seems to be no reason for a specific kind of teachers' CK to bridge the gap between academic discipline and school subject. Bromme (1994), however, argued that the knowledge of the academic discipline and that of the school subject should be distinguished carefully. He, thus, suggested extending Shulman's (1986) model by introducing the two subcategories *content knowledge about mathematics as a discipline* and *school mathematical knowledge*. Hence, Bromme's (1994) model of teachers' professional knowledge takes into account the gap between academic and school mathematics. Regarding the question as to what enables teachers to bridge this gap, Bromme (1994) argued that “the integration of knowledge originating from various fields of knowledge […] is an important feature of the professional knowledge of teachers” (p. 86). However, it is not made explicit, how such integration takes place and what the resulting product of this process is.

The CK that is addressed in the COACTIV study is intended to lie between “the school-level mathematical knowledge that good school students have” and “the university-level mathematical knowledge that does not overlap with the content of the school curriculum” (Krauss, Baumert, & Blum, 2008, p. 876). Therefore, this conceptualization targets the field of tension between academic and school mathematics. However, the released items and their sample solutions suggest that the addressed knowledge is rather close to the mathematical knowledge of the school subject. Similarly, the conceptualization of CK in TEDS-M picks up the distinction between academic and school mathematical knowledge according to Bromme (1994). However, knowledge bridging the gap in between is not taken into account explicitly in this conceptualization and from the released items only very few represent academic mathematics from a university context (Brese & Tatto, 2012).

The conceptualization of CK by the Michigan group (e.g., Ball & Bass, 2003) consists of three parts: *common content knowledge* (CCK), *specialized content knowledge* (SCK) and *horizon content knowledge* (HCK). CCK is defined as “the mathematical knowledge known in common with others who know and use mathematics” (p. 403). Hence, Bromme’s school mathematical knowledge is apparently included in this category. Not entirely clear is, however, what more is encompassed, since very different professions require knowing and using mathematics of some kind, which may in turn be specific to the profession. In particular, one could ask whether the academic mathematical knowledge of a research mathematician is also part of CCK. Taking a look at the released items with which the Michigan group assessed CCK of primary mathematics teachers suggests, however, that CCK was construed as being mainly school mathematical knowledge (Hill et al., 2004).

Contrasting CCK, the construct SCK was introduced as a kind of mathematical CK that is needed specifically for the work of teaching (Ball et al., 2008). Reviewing the authors’ explanations regarding SCK suggests that it may essentially be seen as knowledge needed to bridge the gap between school and academic mathematics: Ball and Bass (2003) emphasized in this context that “knowing mathematics in and for teaching includes both elements of mathematics as found *in* the student curriculum […] as well as aspects of knowing and doing mathematics that are less visible in the textbook’s table of contents – sensitivity to definitions or inspecting the generality of a method, for example” (pp. 8-9). Corresponding to their bottom-up approach, their examples for SCK are mainly concerned with making connections between academic and school mathematics in bottom-up direction: Ball and Bass (2003) explained for instance that teachers have to decide about the mathematical appropriateness of definitions they come across in textbooks. Describing aspects of SCK they also mention that the teachers’ (academic) mathematical knowledge needs to be "unpacked" in order to teach mathematics at school (Ball & Bass, 2003), which can be seen as making connections in top-down direction. The construct HCK was described as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Thus, HCK includes knowledge about the curricular order of contents in school mathematics and their interrelations. Furthermore, the explanations regarding HCK by Ball and Bass (2009) suggest that it also encompasses knowledge about reasons for this curricular structure: "We see that teaching requires a sense of how the mathematics at play now is related to larger mathematical ideas, structures, and principles" (pp. 15-16). Ball & Bass (2009) argue that they "have known from the beginning that there is a kind of content knowledge that is neither common nor specialized" (p. 15). Thus, they indicated that HCK cannot be determined from the perspective of their distinction between common and specialized knowledge. Instead, it may be grasped through the lens of the need to bridge the gap between academic and school mathematics: HCK is apparently a kind of knowledge about interrelations of academic and school mathematics. Hence, although the conceptualization of CK by the Michigan group does not explicitly take into account the distinction between academic and school mathematics, the domains that they uncovered may be nicely interpreted in view of such a distinction. It should, however, be noted that the model was developed for primary teachers and needs to be refined with a focus on secondary mathematics teachers, where academic mathematics plays typically a more important role.

SCHOOL-RELATED CONTENT KNOWLEDGE

So far we pointed out that there is a gap between academic and school mathematics and argued that mathematics teachers need a special kind of mathematical CK about interrelations between these two kinds of mathematics. In this regard, there are promising approaches in mathematics education, but not yet a comprehensive model of secondary teachers' mathematical CK that targets this issue explicitly. Consequently, we introduce the construct *school-related content knowledge* (SRCK) in order to complement CK of school mathematics and CK of academic mathematics. We understand SRCK as a professional CK about interrelations of academic and school mathematics. Informed by early reflections on the relation between academic and school mathematics and the profession of mathematics teachers, SRCK was conceptualized to consist of three facets: (1) knowledge about the curricular structure and its legitimation as well as knowledge about the interrelations between school mathematics and academic mathematic in (2) top-down and in (3) bottom-up direction. In the following, these facets as elaborated in the previous sections will be illustrated by means of sample items that could be used for an operationalization of this construct. In each case the solution is indicated by crosses.

As reasoned above, curricular knowledge in the sense of knowing about the curricular order of contents and their interrelations is not school mathematical knowledge, but concerns the structure of school mathematics on a meta-level. The item in Figure 1 shows an example of how such knowledge may be tapped. For answering correctly it is necessary to know what is involved in justifying that the number π is identical for all circles (interdependences of school contents) and also at which grade level the respective topics are treated (curricular order).

In some German states the number π or an approximation ($\pi \approx {}^{22}/_{7}$) gets introduced already in grade 7 in the context of proportionality (a circle's circumference is proportional to its diameter with proportionality factor π). However, in this school year it cannot be justified that the number π is identical for all circles. Which necessary mathematical topic was not treated yet?

× Similarity

☐ Congruent transformations

☐ Concept of mapping

☐ Limit processes

Figure 1: Sample item “structure of school mathematics”

The real numbers $\mathbb{R}$ can be mathematically constructed from the rational numbers $\mathbb{Q}$ in several ways. Which way of construction is suited as a reduction for the mathematics classroom? Please assume that the existence of examples for irrational numbers was already shown, as usual.

	True	False
$\mathbb{R}$ is constructed from $\mathbb{Q}$ by means of the topological closure.	☐	×
$\mathbb{R}$ is constructed from $\mathbb{Q}$ by means of fundamental (Cauchy) sequences.	☐	×
$\mathbb{R}$ is constructed from $\mathbb{Q}$ by means of nested intervals.	×	☐
$\mathbb{R}$ is constructed from $\mathbb{Q}$ by means of Dedekind cuts.	☐	×

Figure 2: Sample item “top-down direction”

In school, the perpendicularity of two lines is often introduced by double folding as illustrated in the following example. [Picture of a doubly folded piece of paper with perpendicular folding lines]

On which of the following mathematical definitions of perpendicular is this folding instruction based?

	True	False
Two lines g, h are called perpendicular, if they intersect and all four emerging angles are congruent.	☐	×
Two lines g, h are called perpendicular, if $g \neq h$ and a reflection across h maps g onto itself.	×	☐
Two lines g, h are called perpendicular, if h is the bisecting line of the straight angle formed by g.	×	☐
Two lines are called perpendicular, if $g \neq h$ and there is a 90°-rotation that maps g onto h and h onto g.	☐	×

Figure 3: Sample item “bottom-up direction”

Moreover, it was outlined above that teachers need to have knowledge about interrelations between academic and school mathematics in order to teach secondary school mathematics in a way that respects the integrity of academic mathematical ideas. The sample items shown in Figure 2 and Figure 3 illustrate how such knowledge regarding the corresponding top-down and bottom-up direction may be operationalized. Starting with academic mathematical knowledge about different ways of constructing the real numbers from the rational numbers, the item shown in Figure 2 requires knowledge about which of these ideas is compatible with school mathematics in the sense that it can be taught in the scope of the school curriculum. The item shown in Figure 3 addresses an interrelation in bottom-up direction: Answering the question correctly requires knowledge about which mathematical definitions of perpendicular in the sense of academic mathematics lie behind a folding activity described in a textbook. Such knowledge is seen to be needed to understand how the folding can be used to introduce the concept in the mathematics classroom.

In this theoretical contribution, we introduced and illustrated SRCK as a linking construct between academic and school mathematical CK. It is of course an important question whether it is empirically separable from related constructs of professional knowledge, especially academic CK and PCK. The results of a study with 505 pre-service teachers showed that the constructs are indeed separable (Loch, Lindmeier, & Heinze, 2015). The resulting model of the professional CK of secondary mathematics (pre-service) teachers is in particular relevant for designing and investigating secondary teacher education programs and it may also contribute to answering the central question as to what professional CK secondary mathematics teachers need. Moreover, we hope that the construct SRCK provides a new starting point to focus on a professional CK that is on the one hand characterized by a profound understanding of academic mathematics and on the other hand enables teachers to solve the evolving problems of teaching secondary mathematics.

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HOW DO MATHEMATICS DEPARTMENTS EVALUATE THEIR GRADUATE TEACHING ASSISTANT PROFESSIONAL DEVELOPMENT PROGRAMS?

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In the United States, graduate students are involved in the teaching of undergraduate mathematics as discussion group leaders and as instructors. However, little is known about how departments evaluate the quality of the graduate students' instruction or the efficacy of their professional development. We present a mixed-method analysis of a national study that sheds light on both of these topics. We found that graduate students and their professional development are most often evaluated based on student evaluations. Research continually indicates the ineffectiveness of student evaluations as measures of teaching, and so we take this result as a call to develop research-guided evaluation tools for graduate student professional development.

INTRODUCTION

In the United States (U.S.), graduate student teaching assistants and associates (GTAs) play a large role in undergraduate mathematics education (Belnap & Allred, 2009; Ellis, 2014), though typically have little to no prior teaching experience and receive minimal teaching preparation. It is well documented that more rigorous teaching preparation can result in expert-like beliefs, knowledge, and practices (Alvine et al., 2007; Hauk et al., 2009; Kung & Speer, 2009; Luft, Kurdziel, Roehrig & Turner, 2004), making up for the lack of teaching experience of graduate student instructors compared to other types of instructors. In particular, a recent national study found the presence of a robust GTA professional development (PD) program to be characteristic of departments with successful undergraduate calculus programs (Ellis, 2015). Given the need for effective preparation of GTAs in teaching, there is also a growing need to understand how these PD programs are, and can be, evaluated.

Typically a PD program for teachers is marked as successful based on a positive change in teachers' knowledge, beliefs, instructional practices, or their students' success (Sowder, 1997), and thus evaluating a PD program often involves evaluating at least one of these measures. A number of researchers have (separately) assessed multiple K-12 PD programs and determined common characteristics of successful ones (Elmore, 2002; Garet, Porter, Desimone, Birman, & Yoon, 2001; Hawley & Valli, 1999; Kilpatrick, Swafford, & Findell, 2001). These characteristics include, but are not limited to, programs occurring over long periods of time, a focus on content-specific understanding and student thinking and an opportunity for enactment of practices through teaching activities. Currently, there exists no comparable set of characteristics identified as common to successful GTA PD programs. The field of research on GTAs, including research on PD programs, is young and there is still much work to be done.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 227–234. Szeged, Hungary: PME.

Belnap and Allred (2009) noted a need for "research that builds a knowledge base for not just telling us whether a program [of PD for GTAs] had a specific impact, but *why* and *how"* (p. 36, emphasis added). In this report we present findings from a U.S. national survey about what data graduate-degree granting (Master's and Ph.D) institutions are using to determine the efficacy of their GTA PD programs. Two large-scale projects (under the auspices of the Mathematical Association of America (MAA) and funded by the National Science Foundation (NSF)) will provide opportunities to examine the changing state of GTA PD and ways in which departments evaluate these changes.

The work reported on here is the first step in the larger and longer-term effort to understand department change and GTA PD. Here we report on findings from analyses of data from a baseline survey that was designed to provide insights into characteristics of current programs in terms of their content, format and duration. In addition to being a basis for future comparisons, these data allow us to answer the following research question for this study: *How are mathematics departments currently evaluating the success of their GTA PD programs?*

METHODS

Project Background

As further context for this work we briefly describe the two projects and their goals related to institutional change and GTA PD. The first project, Progress through Calculus (PtC) (NSF DUE-1430540), aims to document and facilitate institutional change related to the Precalculus-Calculus II sequence. This project is a continuation of the Characteristics of Successful Programs in College Calculus (CSPCC) study and is specifically focused on chronicling and supporting graduate-degree granting mathematics departments in implementing the characteristics found (in CSPCC) to be related to student success in calculus. As noted above, one such characteristic was a robust GTA PD program (Ellis, 2015). The second project, College Mathematics Instructor Development Source (CoMInDS) (NSF DUE-1432381), aims to support mathematics departments in developing and improving GTA PD programs by broadening access to instructional resources and providing support for individuals and departments in utilizing these resources as well as by connecting researchers and PD providers.

Together, these two projects aim to increase awareness of the need for GTA PD, help institutions learn about different types of GTA PD programs, implement robust GTA PD in relation to other needs of their departments and have the instructional resources to successfully implement such programs. As a first step in documenting and understanding departmental change, the two projects have collaborated to understand the current national landscape of existing programs and the GTA PD-related needs of departments, including the ways departments are currently evaluating their programs. A future step will be to develop evaluation tools for that leverage our findings as well as existing tools and resources from K-12 PD.

Data Collection and Analysis

A survey was sent to department chairs at all graduate-degree granting mathematics departments in the U.S. (n = 341). The survey was designed to document the current status of graduate-degree granting mathematics departments' calculus programs, as well as any changes related to this program. Questions related to GTA PD were jointly designed by members of the CoMInDS and PtC teams.

Department chairs were encouraged to have local departmental experts answer components of the survey with which they were most knowledgeable. For instance, for departments with GTA PD programs, facilitators of the programs would be ideal for answering questions on that section of the survey. The survey was administered using Qualtrics and distributed by the MAA with follow up emails and phone calls to encourage participation. Response rate was 68% (n=223) of all institutions, 75% (n=134) of Ph.D-granting and 59% (n=89) of Master's-granting institutions. For this report we present combined data from Ph.D-granting and Master's-granting institutions. Questions about GTA PD were multiple-choice and open-ended.

Here we discuss responses to a subset of these questions, as shown in Table 1. These questions focus on context of the department's GTA PD program, how they evaluate graduate students in their roles as GTAs, how they assess success of their GTA PD program, and what data they use as evidence in their program assessment. This subset of questions includes both multiple choice questions and open-ended responses questions, asking responders to explain or elaborate their choices to the main questions. In elaborating their selections to the multiple-choice questions, many institutions pointed to specific aspects of their GTA PD program as data for their choices. From these responses we are able to gain insight into how departments currently evaluate their GTA PD programs. We conducted basic descriptive analyses on the multiple choice question data and thematic analyses on open-ended responses to these questions (Braun & Clarke, 2006). Thematic analysis is a bottom-up qualitative approach, where themes are data-driven, though not developed in an "epistemological vacuum" (p. 84). Table 1 shows question numbers, and questions.

#	Question (and multiple choice options)
1.	Is there a required, department-specific teaching preparation program for GTAs in your department? • Yes • No
2.	WHO is the primary audience for your department's GTA teaching preparation program? Mark all that apply. • GTAs who act as graders • GTAs who act as tutors • GTAs who lead recitations • GTAs who are the primary instructor for a course • GTAs who assist with the in-class instruction for a course
3.	Which of the following activities, related to **evaluating** GTAs' teaching, does your program FORMALLY include? Mark all that apply. • GTAs are observed by a faculty member while teaching in the classroom • Student evaluations required by the university or department • Student evaluations are gathered specifically for the purpose of evaluating GTAs (in addition to or separate from the student evaluations required by the university or department)
3e.	• Other (please explain):
4.	How well does your teaching preparation program prepare new GTAs for their roles in the precalculus/calculus sequence? • Very well • Well • Adequately • Poorly • Very poorly
4e.	Please elaborate on your answer above.
5.	Is the department generally satisfied with the effectiveness of the GTA teaching preparation programs currently in place?
5e.	• Yes • The programs are adequate, but could be improved. (please explain) • No (please explain)
6.	What best characterizes the current status of your GTA teaching preparation programs? Mark all that apply. • No significant changes are planned • Changes have recently been implemented or are currently being implemented • Possible changes are being discussed

Table 1: Overview of questions used for analysis and analytic techniques used per question.

RESULTS

We first present results of the descriptive analysis and then results of the thematic analysis. Data from Question 1 indicate the number of institutions that report having a department-specific teaching preparation program for GTAs. Only departments that said “yes” to this question are included in the remainder of analysis. This was done because these departments have influence over the content of the GTA PD program (as opposed to a university-wide program) and can evaluate effectiveness with more insight. Two-thirds of the institutions who responded (148) reported having a department-specific GTA PD program. Of these, data from Question 2 show that the majority (65%) are targeted towards preparing graduate students as recitation leaders or instructors. In the U.S., these are the two main appointments of GTAs: as recitation leaders, they lead discussion sections (also called recitations) for small classes (typically 10-40 students) that also have a lecture component, often taught by a professor or instructor; as instructors, graduate students lead classes as the primary instructor, but often have more supervision than other instructors.

Data from Question 3 provides the frequency of various activities used to evaluate GTA’s teaching. Over 90% of departments with their own program use the university/department-required student evaluations to evaluate their GTAs’ teaching, while about three-quarters use teaching observations by faculty members and one-quarter use additional student evaluations that are specific to GTAs. Note that these percentages do not add up to 100% because responders could choose multiple activities. This finding indicates that student evaluations and teaching observation are used to evaluate GTA’s teaching, but does not indicate what is used to evaluate the GTA PD programs themselves. Findings also show that 57% of respondents report that their program prepares graduate students for their roles well or very well, 66% of departments are satisfied with their programs, and there are no changes underway at 63% of the schools. This indicates that roughly 40% of graduate degree granting mathematics departments in the U.S. are less than happy with the current state of their GTA PD programs. It is these programs that PtC and CoMInDS are targeting and that will be in need of good evaluation tools as they move forward.

Ninety-six respondents provided elaborations for responses to Question 4 (regarding how well the GTA PD program prepares GTAs). Thematic analysis revealed 11 themes in these responses related to what departments use to evaluate their programs. These themes are named and described in Table 2, along with their frequencies. Each response was coded with as many themes as were present.

Theme	Description	Frequency
Student evaluations	Department or university student evaluations used as data to rate GTA PD.	7
Prevented from teaching	GTAs are prevented from teaching if they are not already determined to be prepared. This may be based on performance in teaching a lower level class, being a recitation leader, through an interview, or through practice teaching.	6
Compared to others	The GTA PD program is evaluated in comparison to other departments in the same university or other university, the program is	4
Common Exams	GTAs' students' performance on common exams is used as data to rate GTA PD.	4
Student Grades	GTAs' students' course grades (or pass/fail rates) are used as data to rate GTA PD.	4
Observations	GTAs are observed teaching or leading recitation and these observations are used as data to rate GTA PD.	3
Complaints	The amount of complaints about the GTA is used as data to rate GTA PD.	2
Teaching Award	GTAs' receiving Department or University teaching awards is used as data to rate GTA PD.	2
Other	This included alumni surveys, listening to the advice experienced GTAs give to new GTAs, reviews from faculty, retention of students, and student performance in subsequent courses.	4
Too vague	The response included an evaluation of the program with no data, such as "could be improved" or "is a well oiled machine."	48
Description of program only	The response included a description of the program with no evaluation or data to point to.	28

Table 2: Description of themes from open-ended responses and their frequencies.

As shown in Table 2, 76 of the 96 responses were coded as being either a description of the program without an evaluation, or an evaluation of the program with no data related to the evaluation. Of the remaining responses, the most often used data were student evaluations (7), followed by student performance on common exams (4), student grades (4), comparison to other known programs (4), teaching observations (3), teaching awards (2), and complaints (2). Four responses involved "other" data, including alumni surveys, listening to advice experienced GTAs give to new GTAs, faculty reviews, student retention, and student performance in subsequent courses.

DISCUSSION

The above analysis show that a primary means of evaluation of graduate students as teachers and of their professional development is based on student evaluations. An abundance of research indicates issues with using student evaluations as an evaluative tool (Basow, 1995; Centra & Gaubatz, 2000; Krautmann, & Sander, 1999). Influences on student evaluations include gender bias, preference to easy graders and enthusiastic instructors. Therefore, it is problematic that student evaluation data is still the primary source used to evaluate novice instructors. Relying exclusively on such data may provide incomplete and/or inaccurate information which, in turn, may provide inaccurate information about GTA PD program impact.

The results presented here reveal a number of other measures that mathematics departments are or could use to evaluate GTA's teaching and GTA PD. These include other measures of student performance, such as course grades, grades on common exams, and grades in subsequent courses, as well as more direct measures of teaching performance, such as observations and teaching awards.

Likely reasons for wide-spread use of student evaluations include their ease of use and historical acceptable in the academic arena. Our capacity to improve learning of undergraduate mathematics is tightly linked to our capacity to provide effective instruction. The community would benefit from insights into the effectiveness of GTA PD from robust, research-guided evaluations. Those data-driven insights could then be used to inform mathematics departments' plans for improving the preparation instructors receive for their teaching responsibilities.

Acknowledgement

This material is based in part upon work supported by the National Science Foundation under Grant Numbers DUE-1432381 and DUE-1430540. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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CREATING A SYNERGY BETWEEN MANIPULATIVES AND VIRTUAL ARTEFACTS TO CONCEPUTALIZE AXIAL SYMMETRY AT PRIMARY SCHOOL

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In this work we investigate the potentiality of carrying out synergic activities using manipulative and virtual artefacts for the purposes of constructing/conceptualizing axial symmetry and its properties. The research study is based on the design and implementation of a teaching sequence involving 4th grade students. Both the design and the analysis of data is framed by the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) According to the results of the study the combined, intentional and controlled use of manipulative and virtual artefacts seems to develop a synergy whereby each activity enhances the potential of the other.

INTRODUCTION

As promoted by Geometry Standards at primary school level, in Italy as in other countries, an interesting lens through which to investigate and interpret geometric objects can be offered by transformational geometry. The study of geometric transformations originates from the observation of phenomena and regularities present in real life, but takes on a particularly important role in the field of mathematics, both as a mathematical concept in itself and as a tool that can be used to describe geometric figures. For transformational geometry to be used efficaciously in mathematics, however, it is necessary to be able to correctly mathematize real life observations.

Moreover, we believe that learners' capacity to visualize geometric relationships can develop as they sort, build, draw, model, trace, measure and construct. Active student involvement in the use of manipulatives is, indeed, fundamental in geometry. Such activities develop students' skills in visualizing and reasoning about special relationships. While the use of real objects and manipulative tools has long been known to be useful to support mathematics learning (Sowell, 1989), the advent of the new advanced technologies has enlarged the category of potential manipulatives. Virtual objects have joined real objects, and virtual actions to manipulate virtual objects can now be used, as well as the physical actions of the hands (Laborde & Laborde, 2011). One example of this is the use of the dynamic geometry environment. However, it is also well known that making use of any type of tool does not in itself guarantee that learners are constructing mathematical/geometric concepts.

The Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008) offers an effective reference framework within which to study the relationships among artefacts, the actions they allow one to accomplish, and how pupils using them are constructing mathematical concepts. In the present work, that fits into this research field, a sequence

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 235–242. Szeged, Hungary: PME.

of activities has been created, to be carried out using artefacts of different nature and aimed at promoting the construction/conceptualization of axial symmetry and its properties. The focus is on investigating a synergic use of artefacts. The research hypothesis is that when passing from manipulative artefacts to virtual artefacts and vice versa, a synergic action will develop so that each activity boosts the learning of the others. In this report we present the design and implementation of a teaching experiment carried out with the participation of pupils attending the fourth year of primary school, and analyze some of the interactions that developed in the classroom.

THEORETICAL FRAMEWORK

As stated above, in this study we refer to TSM. The main aspect we focus on is the double semiotic relationship of the artefact with both the task and the mathematical knowledge; such a potential can be used by the teacher as a tool of semiotic mediation in order to make students construct mathematical meanings.

In semiotic activities various signs are produced: the “artefact signs”, that often have a highly subjective nature and are linked to the learner's specific experience with the artefact and the task to be carried out; the “mathematics signs”, in other words the knowledge of mathematics to which the “artefact signs” must evolve; and finally the “pivot signs”, namely semiotic chains that illustrate the evolution between artefact signs and mathematics signs, through the linked meanings.

> Through a complex process of texture the teacher construct a semiotic chain relating artifact signs to mathematics signs, expressed in a form that is within the reach of students. In this long and complex process, a crucial role is played by other types of signs, which have been named pivot signs. […]they may refer to specific instrumented actions, but also to natural language, and to the mathematical domain. Their polysemy makes them usable as a pivot/hinge fostering the passage from the context of the artifact to the mathematics context (Bartolini Bussi and Mariotti, 2008, pag. 757).

Finally, we must underline the importance of organizing the teaching so that it may, during this evolution, foster the collective production and development of signs through Mathematical Discussion (Bartolini Bussi, 1998).

RESEARCH METHODOLOGY

Four tasks involving manipulative artefacts (paper and pins, below denominated P+P) were constructed, and four additional tasks, involving as virtual artefact an interactive book (IB), were designed. The pages of the interactive book were created by modifying a Cabri Elem Activity Book contained in the collection *123... Cabri* (http://www.cabri.com/special-pages/bett2010/). The tasks were proposed to two pairs of children attending the fourth year of primary school. The teaching experiments were videotaped using two cameras, a fixed one facing the pupils and a second one focused on the artefact in use. Conversations were transcribed, that also took into account the specific actions taken with the artefacts. The videotapes and transcriptions were then used to analyse the teaching experiments.

In this paper we will refer only to data coming from the first two of these tasks and we will show, not only the unfolding of the semiotic potential related to each of the two artefacts, but also how a synergy between them can foster the construction of mathematical meanings.

Description of the tasks

In the first task the artefact used was the manipulative one (P+P): a paper with a line drawn on it along which the paper should be folded, and a pin to be used to pierce the paper. In the second task the artefact used was the virtual one (IB): the interactive book based on the dynamic environment Cabri Elem. Task 1 was constructed as follows.

When given a paper with a figure drawn in black on it, a red line is drawn at the moment when the pupil receives the paper and s/he is asked (a) to draw a figure like the black one in red, symmetrically to the red line by folding the paper along the line and using the pins to identify the symmetrical reference points, pressing and then piercing the paper to mark them. After completing this assignment, on the same paper a blue line is drawn and the pupil is asked (b) to draw a blue figure like the black one, symmetrically to the blue line, in the same way as before. Figure 1 shows the paper as it appears at the beginning of this second assignment. In a third assignment (c) the pupil is asked to write, explaining why, how s/he drew the red and blue figures and what looks the same and looks different about them.

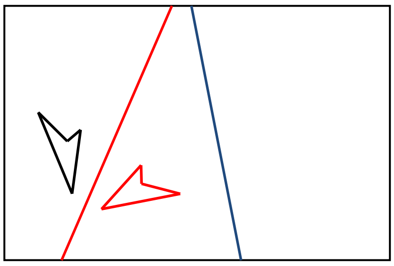

Figure1

Task 2 was constructed as follows. On the first page of the interactive book there was a red line and a point A, and the tools/buttons “Symmetry” and “Name”. The assignment was: “Using the button “Symmetry” construct the symmetrical point to point A with respect to the red line and call it C.”. Clicking on the arrow to continue, the tool/button “Trace” will appear, and then, one at a time, the assignments: “Activate tracing of point A and point C. Displace A. What moves? What doesn't move? Why?”; “Activate tracing of point A and point C. Displace the red line. What moves? What doesn't move? Why”; “Finally, displace point C. What moves? What doesn't move? Why?”. The pupil is asked to write the answers to the questions in a summary table.

Analysis of the semiotic potential and the schemes of use of the artefacts P+P and IB in relation to the described tasks

According to the TSM, we assume that the meanings' construction and their emergence through signs' production is based on the development of utilization schemes related with both the artefact and the specific task (Bartolini Bussi & Mariotti, 2008, p. 748).

The artefact P+P, related to Task 1, evokes three important mathematical meanings: firstly (1.1) the idea of the symmetry axis, expressed by folding the paper along a line; then (1.2) the idea of symmetry as correspondence of points, expressed by the holes in the paper made by piercing it with a pin; finally (1.3) the idea that the symmetrical figure depends on the axis, expressed by comparing what changed and what didn't change in two symmetrical figures when drawn on different lines after folding the paper.

The artefact IB, related to Task 2, evokes the following mathematical meanings: (2.1) the idea of symmetry as a correspondence of points, expressed by clicking on the tool/button “Symmetry”; (2.2) the idea that the symmetrical point depends on the point of origin, expressed by clicking on the tool/button “Trace” for the point of origin and the symmetrical point, and by dragging the point of origin; (2.3) the idea that the symmetrical point depends on the symmetry axis, expressed by clicking on the tool/button “Trace” for the point of origin and the symmetrical point, and by dragging the axis; finally (2.4) the idea that the symmetrical point depends on both the point of origin and the symmetry axis, expressed by dragging the symmetrical point.

We emphasize that as regards the meaning (2.4), in the dynamic geometry environment used, unlike in Cabrì Géometrè, for example, it is possible to drag the symmetrical point obtained, and this in fact allows the whole paper to be “shifted”.

ANALYSIS AND PRELIMINARY RESULTS

Episode 1 – The pupils carried out Task 1. During the following Mathematical Discussion the teacher (T.) asked them to describe what they had done so that she could do it too. The first part of the discussion focused on the choice of points to mark by piercing the paper. Not all the pupils immediately realized that it would have been enough to mark the vertices of the figure, but at the end of the discussion none of them seemed to have any doubts about that. Then Flavia (F.) intervened and, as though demonstrating an important but obvious point, moved her open hand from right to left to simulate the fold, saying “we must fold the paper along this red line” pointing to it with her finger. At this stage the discussion (see Tab1) was concentrated on the use of the pin.

This episode shows the unfolding of the semiotic potential, as expected, but it also illustrates a first evolution towards the mathematical meanings that are the aims of the teaching intervention. The intervention of the teacher is fundamental in inducing the pupils to express the personal meanings, and the different reformulations showed how such meanings evolved from the description of the action to the idea of a correspondence of points.

Transcripts and *gestures*	**Analysis**
F. so with this pin, after folding... we must here… see this point?... we must, how can I say?, pierce it with the pin... *She takes the pin, shows it, gives it to the teacher, goes toward the paper on the desk and with her finger, presses on the point where the pin should be used to pierce the paper.*	When F. indicates the point with her hand, she presses on it as if she wants to simulate the piercing operation. The act of pointing and pressing her finger on the paper is an artefact sign that mimics the piercing act.

Transcripts and *gestures*	**Analysis**
T. like this? Shall I pierce it? (Ita. “punto?”) *She presses the pin on the vertex indicated by F.*	Here the teacher has the paper in front of her, folded along the red line with the black figure facing upward; she asks for confirmation before proceeding, repeating the words used by F. The teacher reflects and accompanies the act of preparing to pierce with the word “point” , a pivot sign because (in Italian) the word (“punto”) invokes both piercing and pressing, as well as the mathematical sign “point”.
F. Yeas, but hard, so that it comes out on the other side	Here it can be seen that pointing and piercing with a pin can evoke the idea of a correspondence between the point of origin and the symmetrical point.
T. Why must it come out on the other side?	
F. Yes, we need the point to come out on the other side ...to obtain the figure, to join the various points and, at the end, make the figure	The point-to-point correspondence obtained by piercing the paper with the pin at the vertices is therefore, in Flavia’s view, the thing that makes it possible to obtain a symmetrical figure. The meaning of correspondence among figures has emerged.
D. …that if we do not make the points… that is, if we do not transfer the points on the other side, it is almost impossible to do it [the figure]	The words “transfer” used by Davide (D.) reinforce the idea expressed by F. that the pin must pierce through the paper to the other side. The sign that emerged in relation to the use of P+P was evolving, thanks to the shared discussion. In fact, “transfer” is a pivot sign, because on the one hand it expressed the action of piercing through to the other side and on the other, the mathematics transformation sign that we aimed to construct through understanding the meaning of symmetry as a correspondence of points.
T. well, so… shall I pierce (Ita. “puntare”) all the four vertices? …and when I have the four points, what I have to do?	After asking all the pupils if they agreed, the teacher pierced the paper at all the points and then reopened the paper.
F. we have, firstly, to identify the points… this one here corresponds to this one… *She points with the index finger of the right hand at a vertex of the black figure and with the one of the left hand at the corresponding point*	The artefact sign “point” has evolved to the meaning of a point-to-point correspondence: F. uses the word “correspond”

Table 1.

Episode 2 - This episode occurred during the Mathematical Discussion that was held after concluding Task 2 in which: the pupils had constructed the symmetrical point (C) of another point (A) with respect to a line, using the virtual environment tool “Symmetry”; they saw what moved and what didn't move while dragging the point A, the symmetry axis and the symmetrical point C. The analysis of video recording of the discussion show two phases. In the first phase we can recognize the unfolding of the semiotic potential of the artefact IB; pupils recognize that point A can be freely moved, but also, thanks to the tool/button "Trace", it become evident for them that point the symmetric point C depends on the point of origin A and on the axis. For example, Flavia says: "I am only displacing A, but since there is a symmetry between the two points, C moves too". In the second phase of the discussion the analysis shows evidence of the possible synergy between the use of the two artefacts, that is how the cross reference to the use of them both, foster the construction of the mathematical meaning of the functional dependency between points in a symmetry. Let us analyse the transcript of this second phase of the discussion (Tab 2). Aurora is dragging point C obtained by symmetry and consequently one can observe that both point A and the symmetry axis is moving. As stated above, the visual effect of dragging C is to cause a shift of the whole set of objects on the virtual paper. When the teacher asks "What moves? And what doesn't move?" Flavia answers, perplexed "It all moves… what doesn't move… is nothing at all!".

Transcripts and *gestures*	**Analysis**
F. when we drag point A, point C moved but the line didn't!... I can't explain it… no, but why should it be normal… but perhaps because C was created by us so… so <u>in the same way as we did with the paper</u>… *With her right thumb up she gestures behind her* ... point A is our black figure,… thanks to the line… since the line moved… first the red and then the blue… so C moved. Now… if C moves… everything moves… why?…	F. has difficulties in understanding why the two points behave differently when she drags them. She mentally reviews the experience gained with P+P and draws a relation when she says "like we did on the paper", associating her words with a movement of her thumb referring to what happened before. She associates the dragging of the line in the interactive book to the two lines, red and blue, that were used for the tasks on paper, synergically linking the meanings acquired during the two experiences. But then she needs to reflect further...
T. let's do this: here is a paper with a line, a point A, and a pin…	The teacher picks up on her mention of the tasks on paper and suggests to go further comparing the two experiences...
Flavia takes the paper, folds it and presses on point A, pierces the paper, removes the pin and reopens the paper F. now we find the point on the other side… these are symmetrical… now let's pretend that A moves here *She folds the paper again, presses the pin on another point on the paper, pierces and turns the paper with the pin still inside. She sees that the pin doesn't come out in correspondence with the previous hole (symmetrical to A) but in a different point and says* so C moves. Now… if I move the line… *She folds again, pierces on A and turns the paper* A doesn't move, but only C does…	Using the P+P artefact F. reconstructs the new situation proposed by the artefact IB. She has no difficulty in simulating the dragging of A, pressing the pin on another point on the paper, and she verifies the effect of this action by noting the position of the new hole, that is different from the previous position. This effect is translated into the sign "it moves". Then in the same way she simulates dragging the line, making a new fold and pressing once more on A. She verifies the effect of this action on the position of the new hole, that is again different from the first position. The effect is translated into the sign "it's moved".

Transcripts and *gestures*	**Analysis**
M. so we can say that if we move A then C will move, if we move the line then C will move…the only one that is dependent is C… that depends on A and on the line.	The word "dependent" used by Morena (M.) reinforces the idea expressed by F. that dragging free objects produces a movement on the dependent objects. The signs that emerged from the synergic use of the two artefacts, P+P and IB, are evolving in the shared discussions. In fact, "dependent" is a pivot sign that on one hand expresses the effect of the dragging action, and on the other the mathematics meaning of functional dependence.
T. Now, can you move C?	At this stage the teacher draws attention back to moving the symmetrical point.
Flavia presses the pin in at point C, without folding the paper, points with her finger to the line she has chosen and then folds the paper and pierces it at point A F. …no because… if I take this C there will surely be A already on the other side… so if I move C… C must have to move the line otherwise the same point will come out on the other side… oh, no! … So if it is necessary to move C it will all have to be moved because it's not possible to move just the symmetrical point!	What F. does at this stage is essentially to reflect on inverse transformation. She starts from the symmetrical point and associates point A to it. She realizes that two distinct points cannot be obtained as the symmetrical of the the same point and so the line must necessarily be displaced. It is an indirect argumentation, extremely sophisticated, based on the functional meaning of the symmetry, on its being univocal and on that each line defines a unique symmetry.

Table 2.

This second episode shows how when using the IB the meanings of the correspondence of points and of the symmetry axis emerge once more. In addition, the excerpt in Tab2, and in particular the final argumentation by F., shows how the synergic use of the artefacts led to a consolidation of the mathematics meaning of functional dependence. A fundamental role is played in the process by the characteristic of this particular virtual environment whereby dragging the point of origin and dragging the symmetrical point produces a different behaviour. This behaviour initially destabilizes the pupils but then it induces them to go back to using the manipulative. Thanks to the teacher's intervention, when she gives Flavia the paper after she had mentally thought back to it, Flavia attempts to transfer the drag action and the trace function from the IB to the P+P, so creating a synergy between the two artefacts. The dependence is then linked not only to the drag movement, that is typical of a virtual environment, but also to a particular way of transferring this movement onto paper. Ultimately, therefore, the meaning emerges most strongly not through the unfolding of the semiotic potential of the two different artefacts, but through the synergy activated by the comparison between the experiences with them.

CONCLUDING REMARKS

In this report we have shown how in passing from the use of manipulative artefacts to virtual artefacts and vice versa, a synergy is created so that each experience enhances the potential of the other. The study is still in progress but the results obtained encourage us to go forward and develop a long term teaching experiment to confirm them. In accordance with the TSM, and in particular with the didactic cycle model, we intend to verify the efficacy of the observed synergy in a longer sequence of didactic cycles.

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THINKING RELATED TO ENACTIVISM AND NOTICING PARADIGM IN MATHEMATICS TEACHER EDUCATION

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This paper focuses on the analysis of data from a Bristol project by both Bristol (enactivism perspective) and Alicante (noticing paradigm) teacher educators to support methodological discussion of samenesses and differences in our approaches. We have found the growth indicators (Jacobs, Lamb, & Philipp, 2010) have helped us to see teachers' changes in noticing children's mathematical thinking. However, the indicators and our discussion of them from our different perspectives have led us to note a significant shift (not captured in the growth indicators) from staying with the detail to a more general label. Our analysis shows that the articulation of this kind of movement supports future noticing.

INTRODUCTION

The Universities of Alicante and Bristol through their education departments are involved in mathematics teacher education. Their staff members are taking part in a long-term collaboration because, in discussions at international conferences, it became apparent that Alicante's use of the noticing paradigm and Bristol's use of enactivism led to interesting overlaps and differences in their practices of teacher education, particularly the way they work with teacher education students.

This paper focuses on the analysis of data from a Bristol project by both Bristol and Alicante teacher educators to support methodological discussion of samenesses and differences in our approaches (the actual work of the teachers and outcomes for the Bristol project are not the main focus). Given that the Alicante group already used the framework of growth indicators, described in a list of six shifts (see next section), we decided to all use that framework for analysis. After discussions, we also decided that we would focus on fewer of the shifts to make the task manageable for the Bristol group (the first four indicators). The objective of our analysis was looking for evidence of any shifts in relation to developing noticing expertise in relation to the first four indicators proposed by Jacobs, Lamb, and Philipp (2010). We would take any such shifts as indicative of teacher change.

We will discuss the Alicante perspective in detail in the next section on 'Noticing and Teacher Change'. There are a number of accessible references to enactivism (ZDM 47(2) Special Issue most recently) but for the purposes of this article enactivism accepts the biological basis of being where knowing is doing. What the Bristol group are likely to focus on in analysis of data is what is being done.

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NOTICING AND TEACHER CHANGE

Learning to notice is part of developing expertise (for an enactivist perspective on developing expertise see Brown and Coles, 2011). Noticing what is happening in a classroom is an important skill for teachers. However, noticing effectively is both complex and challenging. Although the skill of noticing has been conceptualised from different perspectives (Mason, 2002; van Es, & Sherin, 2002), all of them emphasise the importance of identifying the relevant aspects in teaching and learning situations and interpreting them to take teaching decisions. In fact, the noticing skill could be developed by moving from a focus on teachers' actions to students' conceptualisations and by moving from evaluative comments to interpretative comments based on evidence.

Pre-service teachers who make sense of students' thinking gain important insights into how students develop mathematical ideas. Therefore, paying attention to students' thinking encourages teachers to determine what their students already know or do not know, supporting their decisions as teachers. Recently, some research has focused on how pre-service teachers notice children's mathematical thinking providing contexts for the development of this skill (Bartell, Webel, Bowen, & Dyson 2015; Fernández, Llinares, & Valls, 2012; Jacobs et al., 2010; Schack, Fisher, Thomas, Elsenhardt, Tassell, & Yoder, 2013; Sánchez-Matamoros, Fernández, & Llinares, 2015).

In Jacobs et al.'s study, findings indicated that the skill of noticing children's mathematical thinking could be developed, providing growth indicators that can help professional developers identify, provoke and celebrate shifts in teachers' professional noticing of children's mathematical thinking (p. 196). Specifically: a shift from general strategy descriptions to descriptions that include the mathematically important details; a shift from general comments about teaching and learning to comments specifically addressing the children's understanding; a shift from overgeneralizing children's understandings to carefully linking interpretations to specific details of the situation; a shift from considering children only as a group to considering individual children, both in terms of their understandings and what follow-up problems will extend those understandings; a shift from reasoning about next steps in the abstract to reasoning that includes consideration of children's existing understandings and anticipation of their future strategies; a shift from providing suggestions for next problems that are general to specific problems with careful attention to number selection.

In the discussions between Alicante and Bristol, we are interested in the development of teachers' noticing, particularly the development of teachers' noticing of children's mathematical thinking skill.

THE DATA AND ANALYSIS

Data were four audio-recordings of meetings between teachers on a project. In this project, one teacher from each of three schools met five times over an academic year. These were twilight meetings that generally lasted just over an hour and were often attended by a second member of staff from the school. One of the authors (Alf) convened this group and, in between meetings, visited the schools to observe and then lead sessions with the teachers' classes. These sessions had a focus on running activities and class discussion in a way that allowed and supported student creativity. The focus of the group meetings was on teachers sharing the work they had been doing, which included strategies for developing creativity and tackling underachievement. These audio-recordings were transcribed.

For the analysis, the three authors individually analysed the transcript of the first audio-recorded meeting, looking for evidence of the aforementioned shifts (Jacobs et al., 2010). We discussed agreements and disagreements as we shared what we saw as evidence for shifts. Through these discussions we identified common filters to use in looking at the data. Once we had shared our marked up texts and come to an agreement about what constituted evidence, we applied these filters to the rest of the teacher meeting data. We explain, below, what we consider to be evidence for each of the four shifts.

A teacher gives a general strategy description (indicator 1) when he/she identifies a tool or mentions that the problem was solved successfully but omits details of how the problem was solved. If, later on, for example thinking about whole-number operations, the same teacher comments how children counted, used tools or drawings to represent quantities, or decomposed numbers to make them easier to manipulate, we would see a shift into the consideration of "mathematically important details". Teachers may give general comments about teaching and learning (indicator 2), such as, "I learned that it's important to allow students to use different tools to come up with mathematical problem solution" (Jacobs et al., 2010, p. 186). If, later, they make sense of the details of a student strategy and note how these details reflected what the children did understand, for example recognising the ability to count by 2s or the ability to switch between counting by 2s and 1s, we could identify a shift into giving comments specifically addressing the children's understanding. A teacher overgeneralises children's understandings (indicator 3) when they go beyond the evidence provided. For instance, saying, "children understand subtraction and addition — and which to choose when presented with a problem…" (Jacobs et al., 2010, p. 186). This broad conclusion is difficult to justify on the basis of the children's performance on a single problem on which many may have used different strategies. If, later on, teachers make sense of the details of a student strategy and note how these details reflected what the children did understand in specific situations, we would say that there is a shift into linking interpretations to specific details of the situation. Finally, considering children as a group (indicator 4) is another characteristic of over-generalising children's understanding and a shift is indicated by discussion of anything linked to individual understanding.

Seeing the same

In what follows, we show an example of our analysis on two sections of transcript from the beginning and end of the first meeting, through comparison of what each of us saw as evidence of each of the four shifts that we will discuss after the excerpts. Teacher B (one of the teachers that participated in the project) refers to an EAL child, a child for whom English is an Additional Language. This teacher is describing that she had asked Alf to work with her class on fractions. The students had engaged in describing how to give instructions for someone to turn on the spot (which one student did at the front of the class). The students suggested the notation ‘q’ to stand for a quarter turn and ‘h’ for a half turn. They were then writing down different ways of making a full turn (for example, adding 4 ‘q’s).

> **Teacher B (start of first meeting).** So Alf came in and saw me working with them sort of moving around, getting to know the kids a little bit and then since then Alf has come in twice and done some sessions with them but again feeding in from what we’re doing. It’s been really nice for me to sit to one side and take down some observations. One thing that has come out already was the children who maybe seemed less confident earlier in the week with what we’ve been doing, suddenly had all the confidence in the world when you were there. I don’t know whether it was your explanations or your modelling or I really don’t know what it was but they really seemed to come out of themselves.

> **Teacher B (end of first meeting).** I was just looking back at my notes, the EAL child from my class that was coming up to write and spotting the patterns. So, it was really lovely to see. I think what was nice is when it went even further when they started to move on to two terms and some of them saying ‘Oh that makes eight quarters’ and I was like ‘ok!’ So, yes, really surprising. They could hold all that in their heads - I know that we have three quarters add two quarters add three quarters, oh that’s eight quarters, that’s two turns. How they were holding all that in their head, I thought that was brilliant. Really, really good.

Given that the three of us each saw each of the four shifts, the ‘we’ in what follows refers to all three of us. In relation to indicator 1, across the first meeting we see evidence of Teacher B moving to consider mathematically important details (indicator 1). At the beginning of the first meeting Teacher B’s talk is of general strategies and, towards the end, she describes an example of a pattern described by a student. Teacher B, initially, gave general comments about the teaching and learning situation. For example, she referred to the confidence of children before and after Alf’s lesson. However, we can see a shift into comments addressing the children’s understanding (indicator 2). Later in the same meeting, she says “the EAL child from my class that was coming up to write and spotting the patterns” linking a suggestion of the child’s understanding (that she had spotted patterns) with some specific details of the situation (indicator 3), in terms of her writing on the board.

We observe that Teacher B initially focused on children as a group, “One thing that has come out already was the children who maybe seemed less confident earlier in the week with what we’ve been doing, suddenly had all the confidence in the world when you were there…”. Later, in the same meeting, she focused on individual children (indicator 4) in describing the work of the EAL child.

With this analysis, we observe that the teachers who participated in the project showed evidence of shifts related to the way that they notice children’s understandings. The growth indicators used in the analysis above have helped us to see the teachers' noticing changes. However, the indicators and our discussion of them also raised differences in our perspectives leading us to see a significant shift not captured in the framework.

Seeing differently

In the next transcript, Teacher A describes work with Cuisenaire rods. These rods (also made popular by Caleb Gattegno) are wooden cuboids, all 1cm square in cross-section and ranging from 1 to 10cm in length. Each length is coloured differently. One task Teacher A used was to get students to try and 'build' a particular length using other rods.

> **Teacher A (meeting 2).** So, we've got this boy who actually I don't know if you remember M in the first session and he sat, one of the first times when you came in, when he copied and he sat next to A who records really neatly. He didn't know what was going on but he copied how she recorded, as in one number in each box. So, I was, he's copied, he hasn't done anything. But actually from that he's recording on his own and recording in that way which is really nice. So here it was, they could each choose, they chose their own number and practising how many different ways they could make that number using the Cuisenaire rods, so he picked up the yellow. So, we worked out what number that was and it was 'five'. So, then he started building his five-wall and recording it and for him this is amazing. So, he is knowing that it all equals five. He is beginning to see well he's adding them together even though it's not in the 1 plus 2 plus 3.

In this transcript, we see evidence of Teacher A considering mathematically important details although perhaps, as ever, there are more mathematical issues that could be raised. This teacher has given comments addressing the children's understanding, and is not in the realm of giving general comments about teaching and learning. For example, she says *"he picked up the yellow. So, we worked out what number that was and it was 'five'. So, then he started building his five-wall and recording it…he's adding them together even though it's not in the 1 plus 2 plus 3"*. We also observe that this teacher focused on individual children. In fact, she spent some time discussing child "M", addressing the children's understanding (indicator 4).

However, the framework for growth indicators did not provide a useful tool when analysing the following excerpt of Teacher A in the same meeting (meeting 2)

> **Teacher A (meeting 2).** And I think it goes back to that very first session we did when you let J read those numbers because at that very beginning, it's her trying to spot something and other children are spotting and to us it didn't really make any sense. And it is like letting children like M for example going 'I used a pattern, I did two, two, two, two, two' because he's added two every time and just allowing them to say that out and then gradually you see actually through this that they've then actually begun to spot patterns that they can use that are helpful.

All the indicators say that the movement is from general to particular events in the transcript but in this excerpt the movement was from the particular to the general. Indicators 1 and 2 denote a shift from general descriptions to the particular of classroom events. Teacher A, as we have shown, talked about individual children and events. She is generalising from these observations. We read her articulating a general move in her teaching to "letting children speak" about what they notice. The movement is from staying with the detail to a more general label. However, this kind of label "letting children speak" lets Teacher A develop new actions in the classroom based on children speaking and discovering and, supports Teacher A future noticing as we evidence in the following excerpt.

> **Teacher A (meeting 3).** Yesterday, he was in my group and they wanted to investigate three-digit numbers and what happened when you just took away the units and he found out that oh it had to be a 0 because it's still a three-digit number, they were all doing it but because he was working with his own numbers he was just totally engaged and it was his own discovery and on talk chair at the end, he's made this discovery, but he didn't kind of click that most of the other people on his table had also made that discovery, he was like, it's mine, and he then took that into thousands and he'd done 1023 take way three must equal 1020 because I know if I take away the units it's going to be zero and he recorded his 1020 correctly

Articulating this kind of more general label "letting children speak", we see as significant in teacher learning (from an enactivism perspective, Brown, & Coles, 2012). The movement is in the opposite direction since there is a move from staying with the detail to a more general label to what Jacobs et al. (2010) see as "growth" (from general statements to talking about children), yet we believe the articulation of this type of label supports future noticing. The label is an example of a "purpose" (Brown, 2005) that supports the development of new actions in the classroom, linked to the label.

THINKING RELATED TO ENACTIVISM AND NOTICING PARADIGM

We have found the growth indicators used in the analysis above have helped us to see more in the words of teachers. However, the indicators and our discussion of them from our different perspectives (enactivism and noticing paradigm) have led us to note a significant shift not captured by the indicators: Teacher A, from the detail of her talk about students, is able to articulate a more general statement about principles guiding her interactions and teaching. Articulation of such general statements supports future noticing and the accrual of teaching strategies (actions/doing) in the classroom linked to the statement (Brown, & Coles, 2012). In other words, there are times when a movement 'back' (from the detail to a more general statement) along a growth indicator and it is, for us, a mark of teacher development. This kind of articulation is perhaps also an example of what van Es and Sherin label: "making connections between specific classroom events and broader principles of teaching and learning" (2008, p. 245). In Coles's (2012) study of teachers' responses to video, the suggestion was made that if discussion can be channelled into the detail of events by the facilitator then there is the opportunity for later generalisations and interpretations to lead to new labels for events which can potentially capture new insights and support further noticing (similar to van Es and Sherin's (2008) connecting to broader principles).

One issue with the idea of a "growth indicator" is an assumption of a one-way direction of development. When it comes to making general comments about teaching and learning, compared to focusing on individuals, we see our work with teachers more as a cycle than an upward gradient. There is a need to support discussion of the detail of classroom events, as Jacobs at al. (2010) suggest – if discussion begins at a general level then opportunities for learning are limited since the words being used will be categories and labels that teachers already use to think about their classroom. However, we believe that growth is supported by a cycling back, from the particular to the more general, in order to arrive at the succinct articulation of principles or "purposes" (Brown, & Coles, 2012) that can be kept in mind, to inform future noticing and future actions. This is a similar process to the one we have gone through in exploring our different perspectives. It was important to get into the detail of analysing some data. This process allowed us to see differently.

So, while we agree that there is an important movement away from general descriptions, and furthermore that this move often needs to be supported, we see an important next stage in which we arrive at a new generalisation, perhaps in the form of new labels or words for experiences, that support future learning. The process of dwelling in the detail of events and the arising of new syntheses and labels can become a way of being (or perhaps more accurately, a way of becoming) in relation to developing awareness about the processes of teaching and learning mathematics (Brown & Coles, 2012).

There is a need for more research to investigate the proposed movements in noticing – can we gather evidence, for example, that teachers who make new syntheses in a meeting go on to make use of these ideas in their planning and teaching? This data was collected by an enactivist researcher also acting as facilitator at the meetings so was already in the form of asking for more detail of what happened rather than judgements or accounting for actions. It is therefore not surprising that the data revealed some differences to the growth indicators. The way that we collect data, including the role of the facilitator of group meetings in supporting shifts in noticing, will affect our developing sensitivities as researchers.

Acknowledgements

The research reported here has been financed in part by the Project GV/2015/115 of the Conselleria de Educación, Cultura y Deporte de la Generalitat Valenciana and in part by the project EDU2014-54526-R of the Ministerio de Educación y Ciencia (Spain).

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TRAVERSING MATHEMATICAL PLACES

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This paper discusses a case study of grade 9 students who use a dynamic technology to model motion as a graphical approach to the concept of function. We draw on the notion of place (versus that of space), as offered by Nemirovsky, Ingold and Burbules, to analyse the ways that these students understand their mathematical experiences with the tool. We make use of this notion to talk of learning as a matter of creating and traversing mathematical places, striving to avoid representational visions of learning.

INTRODUCTION

In this paper, we present a case study of how some grade 9 students used a dynamic graphical computer-based technology to model motion and study graphs of distance vs. time as an informal approach to the concept of function. The technology consists of a software application, WiiGraph, that leverages two controllers of the Nintendo Wii game console, taking advantage of strategic thinking concerning game play with the controllers. We frame the analysis through the notion of *place* as it has been used and discussed from different authors in relation to learning and thinking (Nemirovsky & Noble, 1997; Nemirovsky, 2005; Burbules, 2006; Ingold, 2011). We have found it inspiring to take this notion to look at the nature of the learning process in a way that aims to trouble dichotomies between external and internal representations, outer and inner, perceptual and conceptual, body and mind, which are still frequently assumed in the literature. In the same time, the notion of place helps us to turn attention to dynamic aspects of the learning situation, including movement and time. In so doing, we also want to contribute to the current discussions about the role of the body and embodiment in mathematical tool use, highlighting how the barriers between the human body and the tool are always changing, being in a continuous process of becoming that shapes the mathematical activity.

Our starting point is the idea of place as discussed by Nemirovsky and Noble (1997), following the work of Winnicott (1971) and his vision of “the place where we live” as the “potential space” where cultural experience and symbolising can be located. The place where we live is an intermediate area of experience between the individual and the environment, with an indeterminate internal/external nature, which grows throughout individual existence. In particular, Nemirovsky and Noble offer the idea of “lived-in-space” in relation to the context of mathematical experience, to refer to the creation of a place that is neither outside or inside the mathematics learner. Most importantly, they claim that “tool-use always take place in a lived-in-space” (p. 127). Starting from this view, we draw on more recent perspectives that use the notion of place in order to widen the vision of mathematical activity with tools as traversing mathematical places, or inhabited spaces, and we use the case of a classroom episode to exemplify this vision.

PLACE VERSUS SPACE: THEORETICAL HIGHLIGHTS

In a very recent work, Nosrati (2015) offers the idea of temporal freedom as key to mathematical thinking. She uses the image of “mathematics as a space”, “a landscape, in which learners are placed. A learner’s understanding and overview of this landscape”, she says, “is constantly being constructed, invented, updated and changed.” (p. 32). Freedom is meant in terms of moving around, seeing where things lie in relation to each other, and so on. Nosrati adds that “If one is only allowed to move on a predesignated path at any given time, expanding one’s overview and understanding of larger parts of the landscape becomes difficult, if not impossible.” (p. 32). For Nosrati, it is not freedom through space per se (as the notion of landscape might indicate), but through *time*. If the image of landscape helps us conceptualise freedom, she argues that the landscape of mathematics is not to be meant as “physical per se”: “in order to get to know it, it is not spatially that students must move, but temporally.” (p. 33). However, we still think that the landscape image might suggest a vision of mathematics as a static, fixed background where learners are ‘arranged’, yet implying an internal/external dualism between the bodies in the classroom and the body of mathematics. For this reason, we propose to enrich Nosrati’s idea of a space where students are free to move temporally, drawing on the notion of *place*. Ingold (2011), for example, challenges the idea of space with that of place, which calls for a space to be *inhabited*, not “occupied” or “filled with existing things” but “woven from the strands of their coming-into-being” (p. 145). For him, the notion of space is too abstract, empty and detached from the realities of life and experience, to describe the world we inhabit. He wants to reject the idea that life has been “installed” inside things (as an internal property of them), restoring “these things to life by returning to the currents of their formation.” (p. 68). Briefly speaking, he reverses what he calls the logic of inversion, for which “the field of involvement in the world, of a thing or person, is converted into an interior schema of which its manifest appearance and behaviour are but outward expressions.” (p. 68). For Ingold, lives “are led not inside places but through, around and from them, from and to places elsewhere” (p. 148): it is through this movement (wayfaring) and the entwinement of the inhabitants’ trails led by their movements that places are constituted. “Every entwining is a knot, and the more that lifelines are entwined, the greater the density of the knot. Places, then, are like knots, and the threads from which they are tied are lines of wayfaring.” (p. 148). It is re-injecting life into space, through movement, and thinking of inhabiting it, that *makes* a space a place, or using previous words, a lived-in space. The qualities of life and movement characterise places *in contrast to locations*: “places are always “for” someone.”, claims Nemirovsky (2005), “There are no places for an inert object because places are constituted by the living bodies for which the place is.” (p. 49). This vision is further refined in Nemirovsky and colleagues (2013), who describe the *lived experiences* of mathematics learners as saturated with feelings and puzzlements, in terms of “the temporal flows of perceptuomotor activities they inhabited bodily, emotionally and interpersonally.” (p. 407). The dimension of time is again essential: “it implies that any perceptuomotor activity is infused with past and future, so that actions pursued at a certain moment cannot be isolated to whatever is physically "there" at that moment. Perceptuomotor activity is always permeated by expectations, recollections, fantasies, moods, and so on." (p. 378). Burbules (2006) also considers the temporal dimension in/of place as important to distinguish it from space: "places emerge, change, and develop diachronically: a space may be a place at one point in time, but not earlier or later; or it may become a different kind of place." (p. 49). For him, a place is a special, important kind of space, where things, memorable *important* things (whether pleasant or unpleasant), happen, "which mark the space as a place ("this is where it happened"). Places become familiar, acclimated to us as we are to them." (p. 49). Both the space and those inhabiting it are changed in relation to each other, and the latter also stand in a different relation to the space, and to each other, for that they are there. A quality of *immersion* always characterises being in a place, supported by elements of interest, involvement, interaction and imagination, which actively shape and change the experience.

In this paper, we propose to study how some mathematics learners come to inhabit their mathematical activity with a specific technology, reconfiguring spaces into places. This helps us to describe mathematics learning as animated by movement and time, striving to avoid discourses that bring up representational visions of learning.

METHOD AND TECHNOLOGY

The case study is part of a teaching-experiment in which a class of grade 9 students has been introduced to modelling motion experiences for a graphical approach to the concept of function through the study of spatio-temporal relationships (see Ferrara & Ferrari, 2015). The experiment lasted for a period of about 3 months, for a total of 9 two-hour meetings. The regular mathematics teacher was present at each meeting, together with the two authors (the researchers, in the following), who constructed the activities for the students and were actively involved in observing and leading them. The experiences took place in a laboratory room offered by the school for scientific work. The students were asked to work according to different modalities: on individual tasks, in groups (of three people each) and in collective discussions led by one of the authors. All the moments were videorecorded and some groups were filmed from the beginning to the end of the experiment. Some technology was used in the experiment. In particular, the one, which is the focus of the study, is an interactive software application, WiiGraph, that has been released by Ricardo Nemirovsky and his colleagues at the Center for Research in Mathematics and Science Education of San Diego State University. WiiGraph uses two Nintendo Wii controllers (Wii Remotes or "Wiimotes") to detect and graphically display the location of users as they move along life-size number lines. A Wiimote is a remote control for playing games with the Wii console, which supports motion sensing capability. Through the use of a sensor bar, WiiGraph is able to capture the position of the two controllers in an interaction space, where embodied exploration can occur. When each controller is suitably directed at the sensor bar, a corresponding circle appears on the screen, indicating that the software is ready to function and to capture the position/distance of the controller from the sensor.

This is the only constraint to commence a graphing session. In our experiment, the display area of WiiGraph (the so-called Graph Form, which contains the control panel plus a graph area) was always projected on an IWB, with the students sitting all around the interaction space, to share experiences and the screen. As WiiGraph starts working, two real time graphs, each associated to one Wiimote (each line coloured differently from the other line, by default one pink and the other blue), appear on the screen. The software provides several graph types, challenges and composite operations for learners to individually and collaboratively explore, including shape tracing, maze traversal, and ratio resolution. The graphs are generated in the graph area according to selected graph type, operations, ranges, time periods and targets. On the control panel, one can also choose other options, for example play, pause or refresh an experience, but even hide or reveal a specific curve. Regarding our case study, we are interested in the maze traversal, which is made available by the Line Graph type plus the choice of the "Make your Own Maze!" option, as we explain in the next section.

MAKE YOUR OWN MAZE!

The activity we consider here asked the students to deal with particular tasks using the "Make your Own Maze!" option of WiiGraph. The Line Graph type, which the students had used before, was selected, for having on the same Cartesian plane two distance versus time lines corresponding to the Wiimotes' movements in front of the sensor bar. Being a and b the positions of the two controllers and t the variable for time, the two coloured lines are the graphs of $a(t)$ and $b(t)$ in a fixed time interval of 30 seconds by default (Figure 1a). Selecting the "Make your Own Maze!" option (MYOM) offered the possibility of having a new graph on the screen as a target and trying to traverse it with $a(t)$ and $b(t)$. The target is built with a certain number of inflection points, width, tension and layout, which determine a degree of complexity for the graph (Figure 1b). The closer the traversal is to the target the higher score a user achieves. The score is a measure of precision over the degree of complexity and appears at the end of the session for both users. The activity challenged pairs of students to get the best score, moving the controllers properly in front of the sensor. This case study focuses on the experience of Lorenzo and Alberto, who were asked to traverse the target graph in Figure 1c (with 42 as degree of complexity). Before them, only another pair of students took part in a challenge with a different target.

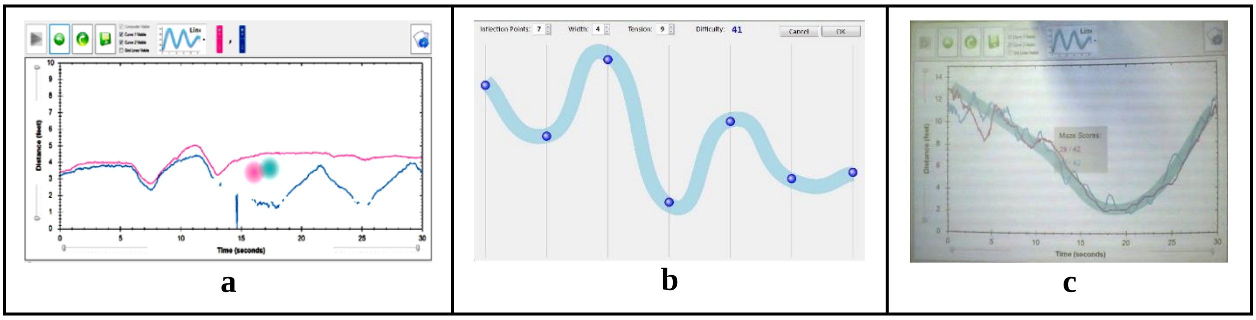

Figure 1. (a) Line Graph lines; (b) MYOM target; (c) Lorenzo and Alberto's target

In the first part of their experience, Lorenzo and Alberto have started to move in front of the sensor, with the target graph visible to all the students on the IWB. Each staring at the screen and holding his Wiimote with two hands, the two students were both rigidly keeping their arms stationary in ways that the Wiimotes remained at the same height during the movement. However, Lorenzo was walking well-standing and keeping his feet close to each other, while Alberto was making big steps and moving back and forth with his torso, protruding himself towards the graph area, with his feet almost still (Figure 2a-b; Alberto is on the left, Lorenzo on the right).

With the first trial, Lorenzo has achieved a score of 31/42, Alberto a score of 16/42. A second part of the experience is constituted of a brief consult with the group-mates, then the two students tried again to traverse the same target graph, with the aim of improving their score. Lorenzo received by his group-mates the suggestion of starting the movement farther from the sensor with respect to the first trial. The pair repeated the challenge, with the students moving again in two different manners, but with the same intensity and effort as before (Figure 2c-d). Lorenzo and Alberto were really attentive to their paces and to maintain their steps as regular as possible in order to improve their previous movement. The final scores corresponded for both to 29/42.

a **b** **c** **d**

Figure 2. Lorenzo's and Alberto's ways of moving to traverse the target graph

The focus of our study is on the third phase of the activity, when the students took part in a collective discussion with their class-mates and one researcher. In particular, we see how Lorenzo and Alberto, in this moment, confront with their ideas about the qualities of movement and the expectations on the target graph, and how a couple of other students intervenes in the dialogue, enriching understanding of the experience. This phase is analysed in the next section, in which we centre on the interesting spaces for embodied kinaesthetic interactions that MYOM offered to the students.

ANALYSIS

As soon as the 29/42 scores appear on the screen, the spread emotion of the class is surprise, and many students pronounce: "Tie!". The researcher asks for impressions by the two players, and the following discussion starts:

(R=researcher, L=Lorenzo, A=Alberto, F=Federico, G=Giulio; L/RH=left/right hand)

1 R: Feelings? None? Disappointed?

2 L, A: No!

3 L: I had understood what to do, the only problem was doing it! (The class laughs)

4 R: Why? What was the very problem in doing it, in trying to do it, for you at least?

5 L: To me, it was just moving at the right speed.

6 R: What do you say, Alberto?

7 A: I had some problems at the beginning, I wasn't able to understand how to stay in the line (*LH index finger pointing at it*), then, I have understood it. But, the difficult thing was remaining at a constant speed so that our graph (*LH index finger pointing at it*) remained in (*Mimes the shape of the target graph with his LH index finger, in the air in front; Fig. 3a*) the graph (*Repeats the previous gesture closer to his torso*) displayed on the IWB.

8 R: Tell me, Federico.

9 F: To me the difficulty is not much, yeah also a bit that, but also looking that, instinctively, when we go under [the target] (*Moves his RH index finger downward in the air; Fig. 3b*), we are instinctively moved to go forward (*Moves forward his RH as though to hold a Wiimote; Fig. 3c*). Indeed, when they were a little out [of the target] (*RH index finger pointing at the screen*), instead of going backward (*RH index finger, pointed at the interaction space, moving from right to left. Gazes at the interaction space; Fig. 3d*) and entering the graph, they were going forward (*RH moving forward again as though to hold the controller*), because one is instinctively moved to go towards the graph (*Repeats the previous gesture*), and instead one is to stop the drive, and to go back (*RH moving towards his torso*), and enter the graph.

10 G: To me, instead, for what I have seen, the difficulty is starting with, well, from the right distance, and also starting suitably doing the graph, because then, seeing from there, they have much got the hang of it. Indeed, in the end, they have chased very well, but at the beginning, it is visible just that they had just to, I don't know how to say

11 R: get the hang of it, get acquainted with it?

12 G: Yes, get the hang of it.

a **b** **c** **d**

Figure 3. (a) "remained in" (A); (b-d) "under", "forward", "backward" (F)

Lorenzo and Alberto immediately introduce in the discourse the question of speed as a crucial, problematic and difficult aspect of their embodied kinaesthetic experience. Lorenzo speaks of “moving at the right speed” (line 5), while Alberto talks about “remaining at a constant speed” (line 7). While Lorenzo feels this as *the* problem he encountered in his *past* movement, for Alberto it is more a matter of *understanding* the challenge, or “how to stay in the line” (line 7). In fact, Alberto makes explicit the connection between remaining at a constant speed and remaining “in the [target] graph” (even actualised through gestures; Figure 3a) with 'their' line (“our graph”), revealing how the traversal of the line on the screen and movement in the interaction space are inseparable. The use of the verb *to remain* for talking of both one type of movement and the other is part of the way that Alberto is *now inhabiting* at the same time the interaction space and the graphical space, of his creatively being (and living) at a *single place* where speed is a quality of the graph as well as the curvature of the graph is a quality of movement. Here, students are imaginatively free to move along multiple trails. Alberto *also* incorporates the presence of Lorenzo into this place, as suggested by his use of “our graph” to refer to the graph(s) on the screen, pointed at many times, that both students obtained through their motion experiences. The two graphs becoming one do elicit the emotional and interpersonal way of living the place by Alberto. Federico and Giulio, who have lived the challenge from their sitting position (“for what I have seen”, line 10), *traverse* the place shared by the two players, also thinking about the difficulty of the task. But attention is shifted from speed to distance. While Giulio feels that the *starting* moment is essential (“from the right distance”, “starting suitably”, line 10), for Federico it is more a question of how to “enter the graph” (line 9), which is related to the height of the graph (“when we go under”). Federico's *entering* the graph, both in words and with gestures and gaze (Figure 3b-d), is another way of seeing (from another trail) the traversal previously expressed by Alberto. Federico *now* focuses on the negative of the photograph took by Alberto, looking at the need of entering the target graph once one has turned up “a little” out of it. For him, the difficulty of being at the right place implies the temporal physical dimension of being *there*, in the challenge, holding the Wiimote with the hand (Figure 3c). This singles out the feeling of moving against instinct, which would move *any* player (“we”, “one”) in the 'wrong' direction (“instinctively moved to go forward”, “towards the graph”, line 9).

CONCLUSIVE REMARKS

The episode above has shown four students' ways of inhabiting at the same time the space where embodied interactions with the tool occur and the space of the screen in which graphs are displayed. The mathematical activity of the students is lived bodily, imaginary and emotionally across the thoughtful situations that each learner shares with others, telling stories about speeds that are curvatures and distances that are heights. It is where the lived-in spaces overlap, the recalled stories meet, the moving students encounter, that spaces become places, even a single place where movement and time constitute the inherent nature of the multiple paths that learner might cover.

From this perspective, mathematics learning is not representing or acquiring schemes. It is ways of talking, doing and feeling traversing old and new mathematical places, in an ever-changing creative move across and around recollections and expectations.

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TEACHER STUDENTS ANALYSE TEXTS, COMICS AND VIDEO-BASED CLASSROOM VIGNETTES REGARDING THE USE OF REPRESENTATIONS - DOES FORMAT MATTER?

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Analysing classroom situations regarding the use of representations is an important competence of teachers. There is hence a need for empirically evaluated assessment instruments. Although the format in which classroom situations are presented might play a role when assessing this competence, evidence about the role of different formats such as text, comic and video is still scarce. Consequently, we developed in this study a vignette-based test with six classroom situations and designed each situation in the three formats text, comic and video. N=162 teacher students were asked to evaluate the use of representations in the six vignettes. The results suggest that the competence of analysing conforms empirically to a one-dimensional Rasch model and that texts, comics and videos are comparably effective for assessment.

THEORETICAL BACKGROUND

Due to their abstract nature, mathematical objects are only accessible through representations, which can stand for them in many different ways (Goldin & Shteingold, 2001). Those multiple representations complement each other by emphasising specific aspects of the corresponding mathematical object without being the object itself (Duval, 2006). However, not only the mere existence of multiple representations in the classroom is crucial, but the way they are dealt with. As multiple representations have to be integrated by learners to develop a sufficiently rich concept image of a mathematical object (Ainsworth, 2006; Lesh, Post & Behr, 1987), changes between those different registers of representations become necessary (Duval, 2006). Changing representations, however, is cognitively complex and often leads to difficulties in understanding (Ainsworth, 2006; Duval, 2006). Consequently, students need to be supported when dealing with multiple representations by encouraging them to actively create connections between different registers of representations and to reflect their use (Duval, 2006; Bodemer & Faust, 2006). Teachers therefore have to be able to identify and interpret elements of classroom situations that are relevant for their students' learning support regarding representations. This can be seen as a prerequisite for reacting adaptively and optimally to the learners' needs (Sherin, Jacobs & Philipp, 2011; Friesen, Dreher & Kuntze, in press). Such analysing of classroom situations requires connecting observations with relevant professional knowledge, which provides in particular criteria for interpretation (e.g. Dreher & Kuntze, 2015; Friesen, Dreher & Kuntze, in press). According to Weinert (1999), specific and context-dependent abilities to cope with professional requirements can be described as professional competences. Analysing classroom situations regarding the use of representations can thus be regarded as an important profession-related competence for mathematics teachers. This is also supported by studies showing that such analysing is an important characteristic of teacher expertise (Dreher & Kuntze, 2015) and is learnable in the context of professional teacher development (e.g. Friesen, Dreher & Kuntze, in press).

Figure 1 shows the analysis of classroom situations regarding the use of representations as a knowledge-based process: the identification of a relevant situation aspect regarding the use of representations is supposed to initiate the process of analysing. In the process, identified situations are critically evaluated and interpreted based on theoretical criteria. The articulation of the analysis results might contribute to the connection between professional knowledge and situation observations as well. The process of analysing is thus not seen as a strictly linear process but may also contain jumps between the steps or even simultaneous processes (Friesen, Dreher & Kuntze, in press).

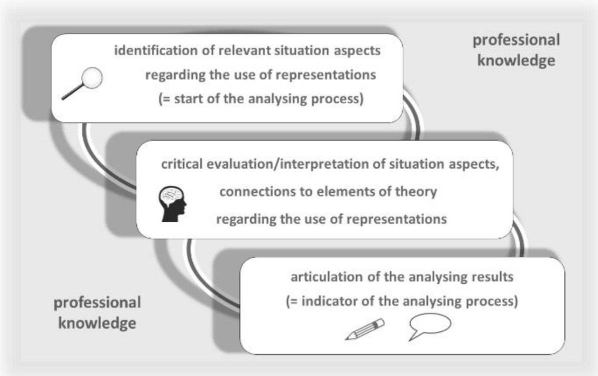

Figure 1: Analysing classroom situations as a knowledge-based process

VIGNETTE-BASED TESTING

When it comes to assessment, the contextualised nature of teacher competence should be taken into account (Weinert, 1999). Accordingly, short classroom sequences called “vignettes” are considered to be particularly suitable to assess competence in close relation to professional requirements of teachers (Oser, Salzmann & Heinzer, 2009). Vignettes can e.g. be realised by using written cases, photo stories or video recordings (Herbst & Kosko, 2013). Many studies argue methodologically for the use of video-based vignettes in order to provide the test takers with meaningful job situations allowing the perception of real-life situations (Blömeke, Gustafsson & Shavelson, 2015; Seidel et al., 2011). Video-based testing has the potential to facilitate knowledge activation and has been shown to be motivating for test takers (Sherin, Jacobs & Philipp, 2011; Seidel et al., 2011). However, studies by Herbst & Kosko (2013) and Herbst, Aaron & Erickson (2013) have risen the methodological question of using other vignette formats than video such as e.g. animations. In a comparison of N= 61 pre-service teachers’ noticing of teacher actions and the accuracy of this noticing, there were no significant differences between the responses to an animation and to a video

(Herbst, Aaron & Erickson, 2013). Keeping in mind the high expense involved in the production of staged video vignettes and video recordings, these results encourage the development of other vignette formats and call for further research.

To describe how various types of vignettes can differ, Herbst & Kosko (2013) propose the categories of temporality and individuality. Accordingly, video vignettes often reproduce the passing of time and preserve the individual features of people and places in the presented classroom events. Texts, however, neutralise individuality and temporality to a high degree by using symbols and expressions such as "the teacher" and expand or collapse the duration of the presented events by providing options like skimming or revising. The position of comics might be somewhere in between: while the individuality of the classroom and the presented persons is reduced to some degree (e.g. boys and girls cannot be distinguished), the facial expression of individual characters can still be observed.

From the perspective of assessing teachers' competence of analysing the use of representations in the mathematics classroom, these aspects might play a role of potential distractors (e.g. added context information in comics and videos have to be processed, the temporality might be an obstacle for analysis) or conversely facilitate analysis (e.g. as context information might support the understanding of the classroom situation). However, to our knowledge, there is hardly any quantitative empirical research which explores the role of vignette formats such as texts, comics and videos for the analysis of classroom situations in a corresponding test instrument. This leads us to the following research interest and research questions.

RESEARCH INTEREST AND RESEARCH QUESTIONS

The research interest of this study is to explore whether the competence of analysing classroom situations regarding the use of representations can be assessed in a vignette-based test instrument taking into account different vignette formats (text, comic, video). In particular, the research questions are the following:

Is it possible to describe the competence of analysing classroom situations regarding the use of representations empirically with one competence dimension using different vignette formats?

Are there differences in the empirical difficulty for different vignette formats (text, comic, video)? Is it possible to identify any systematic patterns?

DESIGN AND SAMPLE

In order to assess the teacher students' analysing regarding the use of representations, we developed six classroom sequences situated in year 6. All classroom situations have a similar structural design showing group work in the context of fractions. Accordingly, each classroom situation starts with the teacher being asked for help by a group of students who have already started to solve a given problem using a certain representation (algebraic or pictorial). The situations were designed by purpose in such a way that the teachers' support of the students is not in line with the theory regarding the use of representations as outlined above. In order to support the students' understanding, the teacher shifts away from the representation the students have already been using and changes to an additional representation. However, this change of representations remains unreflected and unexplained as the teacher fails to connect that additional representation to the representation the students have already been using. Due to the lack of connections between the used registers of representations, the teacher's reaction could potentially lead to further problems in the students' understanding rather than supporting it.

In order to explore validity after the vignette design, the classroom situations were presented to N=5 expert teachers who also teach pre-service teachers in their induction phase. According to their ratings, we selected six sequences for the test instrument in which the support given by the teachers was identified as potentially impeding for the students' understanding due to unexplained changes of representations. In addition, the experts rated those classroom sequences as highly authentic and representative for dealing with representations in the content area of fractions.

To investigate the teacher students' responses to different vignette formats, we implemented each of the six classroom situations as text, comic and video vignettes (see Figure 2). The texts were used as blueprints to design the comics and the comics provided the storyboards for the video recordings. In order to avoid dependencies between the video vignettes, each video was recorded in another classroom showing six different teachers and learning groups. After editing the video recordings, we adapted the comics and the texts, so that the conversations in the classroom situations would have the same wording in each vignette format and the representations used by students and teachers would look similar.

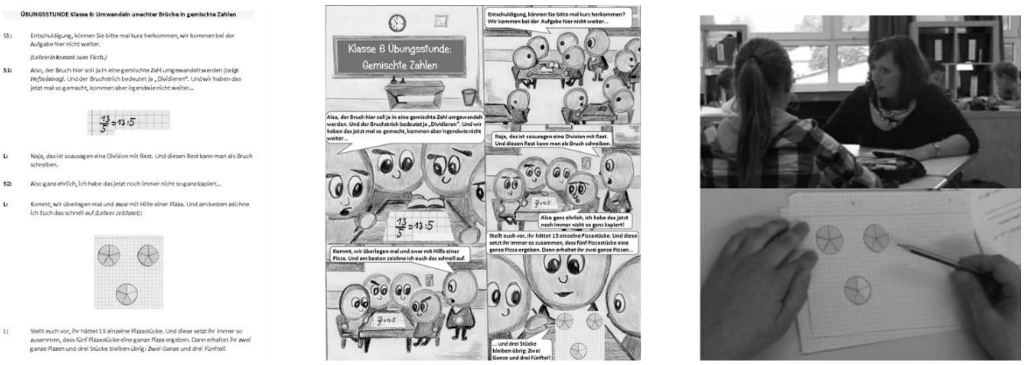

Figure 2: Vignette # 5 as text, comic and video; comic drawn by Juliana Egete

The sample of this study consists of N=162 mathematics teacher students (66.9% female; M_{age}=21.55, SD_{age}=2.38). As they were at the beginning of their professional education ($M_{semester}$=1.80; $SD_{semester}$=1.40), they formed a satisfactorily homogeneous group regarding the progress of their studies. In order to assess their competence of analysing regarding the use of representations, we asked them to evaluate the teachers' support in the six classroom situations responding to the following open-ended item: *How appropriate is the teacher's response in order to help the students? Please* *evaluate regarding the use of representations and give reasons for your answer.* The different vignette formats were randomly assigned to the test takers, so that every teacher student would respond to two texts, two comics and two video vignettes. The videos lasted about 1.5 minutes each and could be paused or watched several times.

DATA ANALYSIS AND SELECTED RESULTS

The answers of the teacher students were coded by two independent raters reaching a good inter-rater reliability with κ=0.85 (Cohen's kappa). The top-down coding scheme was derived from the theory regarding the use of representations as outlined above. Table 3 shows the code descriptions and sample answers to classroom situation number five (see also Figure 2). The distribution of the three codes (see Figure 3) show that only in 25.1% of the teacher students' answers the unexplained change of representations was identified and interpreted as potentially problematic (code 2).

code	description	sample answer
0	refers only to representations used by the teacher without making any connections to the students' question/representation	*"The representation (pizza) is easy to understand and can also be used in other contexts." (#1/20/v5)*
1	refers to representations used by both students and teacher; does not mention that the unexplained change of representations might be problematic	*"The teacher's explanation and representation illustrate the students' problem and the solution very well and helps the students to understand." (#1/1/v5)*
2	refers to representations used by both students and teacher; mentions that the unexplained change of representations might be problematic	"*There is no explanation why five pizza slices form one pizza. The teacher explains how to solve the problem but she does not respond to the students who have already started dividing 13 by 5." (#4/14/v5)*

Table 1: Sample answers and application of codes

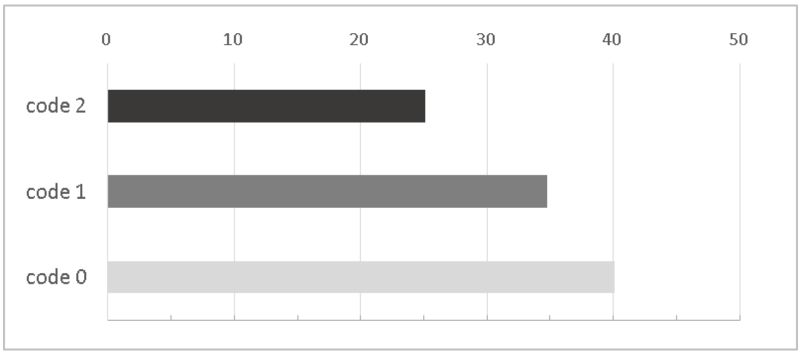

Figure 3: Teacher students' answers: distribution of codes, relative frequencies

Considering the vignette formats, the results of a chi-square test revealed no significant association between the format of the vignettes (text, comic, video) and the teacher students' analysis regarding the use of representations in the represented classroom situations ($\chi^2(4) = 7.09$).

Contributing to the research questions of the study, we applied a Rasch model to the data. In order to reflect the coding of the answers (see Table 1), we used a partial credit model (PCM) which is considered to be highly applicable when partial marks are awarded in an ordered way, each increasing score representing an increase in the underlying ability (Bond & Fox, 2015). In line with that, we took the six vignettes in the three formats as one item each, resulting in 18 items altogether and applied the partial marks as shown in Table 1. The Rasch analysis revealed good fit values for all 18 items ($0.91 \leq$ wMNSQ ≤ 1.16; $- 0.6 \leq T \leq 1.0$) indicating that they sufficiently fit the Rasch model (Bond & Fox, 2015). Related to the first research question, these results suggest that the Rasch requirement for unidimensionality holds up empirically as each item contributes in a meaningful way to the competence of analysing being investigated. The EAP/PV reliability appears to be rather low (0.45) which might be due to the comparatively small number of items (Bond & Fox, 2015).

Related to the second research question, we investigated the association of item difficulties and vignette formats. The Wright map (see Figure 4) shows both persons and items located on the same map with the highest values located on the right of the logit scale and the lowest values located on the left.

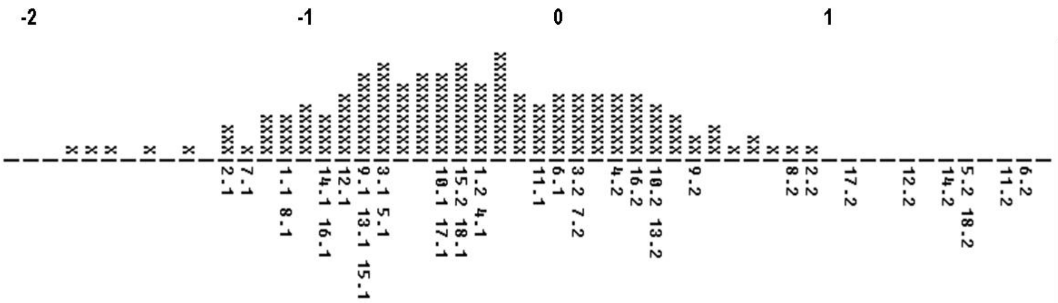

Figure 4: Wright map of the Rasch model (PCM)

As the items were scored polytomously (codes 0, 1, 2; see Table 1), two difficulty thresholds are plotted in the map: above threshold estimate 1, scoring code 1 is more likely than scoring code 0 and above threshold estimate 2, scoring code 2 is more likely than scoring code 1 (Bond & Fox, 2015). The item difficulty estimates show for all 18 items that the step between code 0 and 1 is easier (mostly negative logit scores) than the step between code 1 and 2 (mostly positive logit scores). The person distribution does not extend beyond the range of the most difficult thresholds, which is in line with our expectations regarding the sample of teacher students who were still at the beginning of their professional education (see Figure 4, compare to Figure 3). As the difficulty estimates can be interpreted as interval data (Bond & Fox, 2015), we carried out an analysis of variance in order to investigate the association of item difficulties and vignette formats. In line with the results described above, the comparison of texts (items 1-6), comics (items 7-12) and videos (items 13-18) again did not reveal any significant differences between the three vignette formats (F=0.047, df=4; p= .996).

DISCUSSION

In this study, we intended to explore if format matters in the assessment of a competence which we expected to be unidimensional. The results suggest that when teacher students' competence of analysing the use of representations is assessed with vignettes in text, comic and video format, the test items fit to a one-dimensional Rasch model without exception. As the analysis of the data did not show any significant differences neither with respect to the codes of the student teachers' answers nor concerning the estimated item difficulties, the results suggest that in this study the competence of analysing could be measured independently from the different vignette formats. Consequently, the vignette formats text, comic and video appear to be comparably effective to tap teacher students' competence of analysing regarding the use of representations. These findings may be interpreted as evidence supporting the validity of the test instrument: they suggest that the teacher students' competence of analysing was not influenced by item design factors such as temporality, individuality or the context information that were implemented to different degrees in the three vignette formats. In addition to that, our findings are in line with studies comparing e.g. animations and videos carried out by Herbst et al. (2013), contributing to an external validation of our study. Bearing in mind the high expense involved in the production of staged video vignettes and video recordings, the findings encourage further research on the development of alternative vignette formats in order to assess professional teacher competencies. Moreover in further research, pre-service teachers at an advanced level and in-service teachers should be taken into account as they might perceive the vignette formats in a different way due to their different professional knowledge and teaching experience.

Acknowledgements

This study is supported in the framework of the project EKoL supported by the Ministry of Science, Research and the Arts in Baden-Wuerttemberg.

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SHARED OBJECT AND STAKEHOLDERSHIP IN TEACHER-RESEARCHER EXPANSIVE LEARNING ACTIVITY

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Cultural historical activity theory (CHAT) perspectives are used to shed light on an extended teacher-researcher collaboration, at a Grade 4-6 school in Sweden. Beginning with participant observation and emerging forms of engagement like co-authorship of research reports, the collaboration is understood as expansive learning activity. Treating the practices of teaching and research as distinct yet collaborating activity systems within this, provides an opportunity to analyse the manner in which joint conduct of project related instructional interventions became shared object. This also enabled teacher and researcher to become active stakeholders in each other's practice. Dialectical realisation of stakeholdership and shared object led to reconceptulaisation and transformation of the very horizons of our work.

INTRODUCTION

In his plenary address at the 35th annual PME conference, Konrad Krainer (2011) sought for teachers to become stakeholders in mathematics education research and researchers to become stakeholders in classroom practices, not only to allow for mutual trust but for respective knowledge bases to overlap in reflective rationality. As teacher and researcher, in this paper we shed light on one such instance of stakeholdership, elaborated elsewhere as a case of expansive learning activity (Gade, 2015). Such activity elaborated upon in following sections, exemplifies our collaboration, first with a six month pilot at a Grade Six classroom, followed by year-long project related work to promote communication in mathematics at Grade Four. During this time, our initial engagement as teacher and researcher observer gave way to analysis of intervention related data as well as co-authorship in scientific reporting, such as this research report. Enabling us to build trust and share knowledge over time (Krainer, 2011), such engagement exemplifies the motive, purpose or object of what Engeström (2001) identifies as expansive learning activity:

> The object of expansive learning activity is the entire activity system in which the learners are engaged. Expansive learning activity produces culturally new patterns of activity. Expansive learning at work produces new forms of work activity. (p. 139)

While project related interventions formed the central backbone of our collaboration, in this paper we step back and reflect on the evolving nature of the object we shared in our extended collaboration. Such an eventuality enabled us to work across school and university confines, our respective practices of teaching and research and become stakeholders. In what manner did the shared object in teacher-researcher expansive learning activity, allow for stakeholdership in teacher-researcher collaboration?

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 267–274. Szeged, Hungary: PME.

Prior research points to poorly understood relationships between practices of teaching and research. Writing to the theory-practice issue which remains problematic to this day, Elliott (1991) speaks from action research traditions and points out that teachers feel threatened by theory, produced by outsiders who claim to be experts. Such theory, couched in generalised terms, denies teachers their everyday experiences. From teacher education research, Cochran-Smith (2005) has more recently argued in favour of building a theory for social change by university-based researchers drawing on educational scholarship, while collaborating with school-based teachers who are activists. Within mathematics education research, Schoenfeld (2013) has spoken to the paucity of studies which detail classroom ecologies, while also addressing major problems of practice. Laying down principles of action that could ensure success in mathematics for all, Leinwand, Brahier and Huinker (2014) articulate professionalism of teachers in terms of their ability to enter into partnerships with knowledgeable others, so as to question the existing status quo. Speaking from lesson study research Corcoran (2011) shows how lesson plans say, could be viewed as boundary objects in communities of practice in which teachers needed to constantly "become". Engaging with these issues, in this paper we treat our individual practices of teaching and research as two separate yet collaborating activity systems, with the shared object of each being our joint conduct of project related instructional interventions.

THEORETICAL UNDERPINNINGS

In his version of cultural-historical activity theory, also known as CHAT, Engeström (2001) extends the Vygotskian premise that the human mind develops with meaning mediated by cultural artifacts acting as tools or instruments. To overcome the divide between the Cartesian individual and prevalent societal structures, Engeström forwards a triangular activity system as analytical unit, which incorporates a selection of societal elements. While other scholars have sought for explicit inclusion of emotion while studying human development (Roth & Lee, 2007), and dwelt insightfully on its transformative aspects (Stetsenko, 2008), in this paper we take Engeström's activity system as point of departure. We analyse teacher-researcher collaboration in terms of two separate yet collaborating practices of teaching and research, which we consider as activity systems, whose realisation over time led to expansive learning activity (Gade, 2015). In understanding the shared object of such expansive activity, one which led to stakeholdership in our collaboration, we draw upon five principles laid down by Engeström (2001) which underpin his analytical framework: (1) that an activity system in its network of relations with other activity systems, be treated as unit of analysis; (2) that activity systems be conceived as multi-voiced, incorporating views, traditions and interests of the wider community; (3) that development of activity systems be studied historically, over lengthy periods of time; (4) that the role of contradictions and structural tensions between activity systems, be studied as the source of change and transformation and (5) that the object and motive of expansive learning, with their qualitative transformations, be understood in terms of how radically new horizons and modes of activity are reconceptualised.

In line with Engeström we represent the practice of teaching and research as two triangular activity systems, as in Figure 1. This schematic also depicts collaborative activity between Lotta (as Charlotta is known) and Sharada with respect to our joint conduct of instructional interventions to meet with Lotta's project related aims. The evolving nature of either activity system over time, led us to share project related aims, reconceptualise the very horizons of teacher-researcher collaboration and allowed us to become stakeholders in each other's practice.

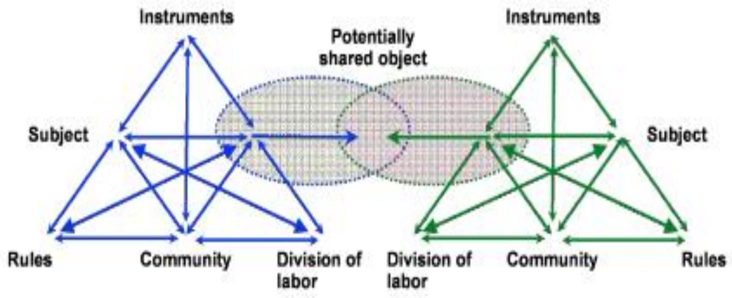

Figure 1: The practices of teaching and research depicted as two collaborating activity systems, with their object being shared (Engeström, 2008, p. 4)

Following non-dualist perspectives of CHAT, we conceive the practices of teaching and research as a network of relations, in both independent and collaborative ways. In line with this view, Lotta was subject in her own activity system for which she utilised various physical and intellectual artifacts to mediate teaching within her classroom (e.g. the textbook, her pedagogy). Directed primarily at students her activity system drew on her practical action or praxis and her practical wisdom or phronesis (Gade, 2014). Sharada likewise was subject in her own activity system, for which she deployed artifacts in educational research (e.g. academic literature, analytical perspectives). Directed at both Lotta and her students, her activity drew on conducting disciplined inquiry within research. In Table 1 below we outline both independent and commonly shared aspects of each activity system.

Activity system	Practice of teaching	Practice of research
Instruments	Of classroom	Of research
Subject	Teacher	Researcher
Object	Conduct of project related instructional interventions	
Outcome	Sharing of object and stakeholdership	
Rules	Of praxis, phronesis	Of disciplined inquiry
Community	School students	Students and their teacher
Division of labour	Primarily of teaching	Primarily of research

Table 1: Comparison of the practices of teaching and research, conceived as activity systems, in the study of teacher-researcher collaboration

CHAT perspectives conceive the purpose or object of any activity system to be their motive as well (Leont'ev, 1978). Our conduct of instructional interventions was both object and motive for each of our activity systems, guided by Engeström's (2001) five principles of expansive learning activity. In deploying two activity systems in our analysis, we investigated the nature of relations which constituted our ongoing functioning, besides the wider network of relations that we were able to enter into (principle 1). These activity systems were far from water-tight and responded to inputs received by way of opinions and interests of both Lotta and her students (principle 2). With both activity systems being multivoiced, we were able to take on each other's role while ensuring democratic participation for all. Achieving multi-voicedness in our study enables this paper to shed light on how our activity systems evolved gradually over time (principle 3) and became expansive learning activity (Gade, 2015). Such conduct provides our study detailed understanding of classroom ecologies (Schoenfeld, 2013) and knowledgeable partnerships which have potential to question the existing status quo (Leinwand et al., 2014).

Rather than personally experienced conflicts, CHAT perspectives treat contradictions as tensions between activity systems and the drivers of societal change and human development (principle 4). For example, in the first of three instructional interventions, Sharada's presence as researcher in Lotta's classroom led us to rectify the faulty use of the mathematical '=' sign by Lotta's students (Gade, 2012). Towards the end of our collaborative work, Lotta likened her experience to a professional development course conducted in her classroom, in contrast to her earlier experience with researchers visiting her classroom as spectators, providing her teaching with little insightful feedback. Through promoting students' development in line with CHAT, our interventions were able to overcome the realisation of a generalised manner of theory which denied teachers their everyday experiences (Elliott, 1991). Our interventions drew on educational scholarship and contributed to wider social change (Cochran-Smith, 2005). Our conduct of project related interventions can also be seen as responding to tensions faced in Swedish society in relation to falling educational standards as reported in International tests, since Lotta's project was one of many funded nationwide by The Swedish National Agency for Education. These societal events provided fillip to the teacher-researcher collaboration and enabled us to respond to local needs in Lotta's classroom, wherein we reconceptualised prevalent pre-existing norms and relations (principle 5). Our need to report on our project related work then led us to enter into and utilise new forms of engagement and work, which characterised our expansive learning activity (Engeström, 2001). Born from our motives within teaching and research, such activity resulted in our crossing the institutional confines of school and university. The newer modes of engagement which we realised were neither clear to us prior to entering into collaboration, nor predetermined in any manner. Our realisation of expansive learning activity was an open ended exercise which drew on our joint conduct of project related instructional interventions, which *became* the object, purpose and motive of our extended teacher-researcher collaboration (Leont'ev, 1978).

METHODOLOGY AND METHODS

Our collaboration extended across a pilot conducted by Sharada in the first half of 2009 in Lotta's classroom and a year-long project during 2009-2010 for which Lotta received funding (Dnr 2009:406). We consider the CHAT methodology that underpinned our study to be tool-and-result (Newman & Holzman, 1997). Unlike designated tools used for obtaining specific outcomes, in this approach a researcher accompanies any subject's use of tools in activity to draw inferences on the human development possible. Recognised as *developmental education* (van Oers, 2009), classroom interventions in line with this methodology, include teachers and also peg instruction to lead, advance and proceed ahead of students' development within instruction. Conducting our instructional interventions in line with this approach, we drew also on the CHAT theory of explicit mediation wherein students' participation in activity was not invisible, internal and implicit; but spoken, audible, visible and made explicit within instructional activity (Wertsch, 2007). Detailed at length elsewhere, these included Lotta's students overcoming their faulty use of the mathematical '=' sign (Gade, 2012), their posing of mathematical problems by making use of textbook vocabulary (Gade & Blomqvist, 2015a), and their use of talk to explore their current understanding of everyday measures, leading also to articulating their nascent and emerging theories of measure (Gade & Blomqvist, 2015b).

Each of our interventions, aimed at promoting students' communication as they learnt mathematics, was conducted at timely breaks within Lotta's teaching to limit their strain on her curricular routines. They were also conducted with students grouped in pairs or dyads. We collected empirical data in the form of students' inscriptions, Sharada's field notes of classroom proceedings and audio recordings of Lotta's whole classroom instruction. It was thus possible for us to carry out multiples levels of triangulation between our three data sources, as well as draw on our experiences of conducting the three interventions as teacher and researcher. This approach informed our analysis and prepared the ground for our scientific reporting of each intervention. The historical progression of such manner of collaboration provided the ground for instructional interventions to become the shared object of both of our activity systems, as further outlined below. In doing so and in line with CHAT, rather than limiting ourselves to methodological individualism, we understood the human mind as actively taking part in ongoing events and practices, geared towards realising specific end products, within instruction, in a non-dualistic manner.

SHARED OBJECT IN EXPANSIVE LEARNING ACTIVITY

To outline the manner in which our conduct of instructional interventions *became* the shared object of teaching and research as activity systems, we trace our teacher-researcher collaboration from inception beginning with Sharada's pilot study in Lotta's Grade Six. In this study Sharada invested in one-to-one relationships with Lotta's students, in order to examine students' narratives as they went about learning mathematics. Upon observing satisfactory realisation of these within her classroom, Lotta began sending students who had completed their assigned classroom tasks, to work at puzzles which Sharada had at hand. Soon after, Lotta requested Sharada to work with a student who was weak, with the consent of the student's mother. The history of these events exemplifies the manner in which we built trust, the first step in the expansive learning activity. The manner in which Lotta acted on this trust leads us to the second step, exemplified by her beginning to take Sharada's presence and input for granted while applying for project funding whose aims were those she thought appropriate as a teacher. We argue these actions mark the nascent beginnings of what became our shared object of collaboration with Lotta's Grade Four students in the year ahead. This sharing took root in yet another pilot in which Lotta conducted an intervention based on the CHAT theory of explicit mediation (Wertsch, 2007). Such conduct had two benefits. First, Sharada was able to study the implementation of CHAT theory within Lotta's instruction. Second, Lotta sought and read the research literature on which the pilot was designed and implemented. Such a theory/practice CHAT approach became our bedrock for conducting further interventions.

Singled out as a contradiction earlier on, Lotta's seeking Sharada's expertise to rectify her students faulty use of the '=' sign, made our collaboration gain agency. In line with action research perspectives, we drew upon our reflexivity to design, conduct and sustain a four-stage action cycle in which Lotta's students offered mathematically appropriate statements (Gade, 2012). Aiming for students to pose mathematical problems as well as reflect on written language, we next had them use vocabulary we chose at random from their textbook. Lotta participated in pair work, standing in for a student who had swimming lessons, and also conducted blackboard work for students to discuss vocabulary which they thought was utilised within mathematics (Gade & Blomqvist, 2015a). Our final intervention was in Lotta's teaching of the topic of measurement. In line with the project aims, we had students use talk to explore their understanding of everyday measures, articulating their nascent and emerging theories of measure. Even as Lotta conducted this intervention by choosing a pedagogical category she thought was appropriate, it was our joint transcription of Sharada's audio-recording and its subsequent analysis which lent itself to our reporting of the landscape study (Gade & Blomqvist, 2015b). In recognising the greater role Lotta began to take on in the trajectory of research being conducted, we point out that while Lotta was anonymised in the first reporting, by the second and third Lotta having contributed to analysis and interpretation of data, was co-author of its content.

From initial steps of our building mutual trust, the process we outline here sheds light on how our teacher-researcher collaboration *became* expansive learning activity. In line with Engeström (2001), the project-wide activity of teacher-researcher collaboration was motivated by our conduct of project-related instructional interventions. Towards the same we utilised new patterns and forms of work, from building trust, interpreting data, to co-authorship. The conduct of project related instructional interventions was also the object to which teaching and research as activity systems were subordinated (Leont'ev, 1978).

TEACHER-RESEARCHER STAKEHOLDERSHIP

We conclude by juxtaposing two aspects which inform the notion of stakeholdership which Krainer (2011) sought, so that our knowledge base as teacher and researcher could engage in reflective rationality. The first is, wider research in education which seeks it's realisation in instructional practices. The second is, CHAT research which allows for its realisation by attending to human development. The need for teachers and researchers to realise stakeholdership is informed from many fronts - to meet demands that teachers have in their everyday (Elliott, 1991), to build theory for social change (Cochran-Smith, 2005), to grasp classroom ecologies (Schoenfeld, 2013), and to question prevailing status quo (Leinwand et al., 2014). In realising stakeholdership in our study and speaking to how these objectives have potential to be met via CHAT perspectives, we point to it's non-dualistic approach which seeks the study of the human mind as culturally and historically situated, distributed or networked in wider society. This premise led to our treating the practices of teaching and research as interacting activity systems (Engeström, 2001), whose motive and purpose lay in the kind of object that was being immediately pursued (Leont'ev, 1978). This involved adopting a tool-and-result approach in our extended conduct of collaborative research (Newman & Holzman, 1997), wherein we conceived teachers as partners and pegged instruction to advance students' development (van Oers, 2009). Such a stance leads us to our next point, that a focus on human development entails that researchers work with various stakeholders in concrete instructional realities and guide the progression of outcomes which may not be envisaged by anyone beforehand. In such conduct the growth of the shared object of the activity systems which collaborate, has potential to overcome contradictions and bring about instructional change.

Finally, we consider our collaboration to exemplify how instructional interventions *became* the shared object of teaching and research. We found that our pursuance of this shared object evolved dynamically and resulted in our *becoming* stakeholders in each other's professional practice. Dialectical realisation of both these aspects enabled us to reconceptualise our existing relationships and radically transform the status quo, besides the historical reality and very horizons of our work.

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GENDER DIFFERENCES AND DIDACTIC CONTRACT: ANALYSIS OF TWO INVALSI TASKS ON POWERS PROPERTIES

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The results of standardized tests such as PISA and the Italian INVALSI, point out the existence of a gender gap in mathematics. This gap is deeply studied in mathematics educations literature. In this paper we analyse two INVALSI items of grade 10 in which male and female answers have distinctly different behaviour. Our aim is to observe if this different trend of male and female answers is influenced in particular by effects of didactic contract. In this analysis we integrate quantitative and qualitative methods. The quantitative analysis is based on IRT models and it allow us to highlight the trend of the correct and wrong answers, distinguishing between male and female. The qualitative analysis involves interviews to students and confirm that the choice of a particular response is influenced by didactic contract effects.

INTRODUCTION

The INVALSI tests are national standardized tests administered every year in different grades of primary and secondary schools in order to have systematic checks on students' knowledge and skills in maths and Italian. The increasing importance given to standardize tests such as INVALSI and PISA, provides new opportunities not only in the evaluation of educational systems' performances, but also in the educational field. If 10 years ago the usage of PISA results in mathematics education was still limited (Sfard, 2005), in the recent years, many researchers began to use standardize assessments for their studies. For instance, the results of PISA and INVALSI tests showed the existence of a gender gap in mathematics in favour of male and gave the opportunity of study this issue in large populations using also specific statistic tools. In our analysis, we observe that the gender gap is not uniformly distributed on all the items of a test: only some of the tasks present a marked gender gap (in terms of percentage of correct answer of male and female). Moreover, according to recent studies on INVALSI tests (Cascella, 2015), the psychometrical analysis of the item functioning reveals that some items present different performances for male and female.

In this paper, we focus our attention on two INVALSI items of grade 10 in which male and female populations have a strictly different behaviour. We select these two items also because we suppose that two wrong choices (we are dealing with multiple choice tests) are related with didactic contract effects. Our purpose is to investigate if, in this particular case, the gender gap and the different behaviour revealed in the quantitative analysis, can be influenced also by a different response to didactic contract for male and female. The first part of our study is a quantitative analysis of the two items based on Item Response Theory models and evidences the different trends related to gender.

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The second part is purely qualitative and consists in interviews with the purpose of understanding the processes that led students to choose a particular distractor.

THEORETICAL LENSES

In the recent years, national (INVALSI) and international (such as PISA) assessments pointed out that in mathematics male and female have different performances: boys outperform girls at all school levels and in almost all the countries. This issue has been debated for several years and a large number of studies has focused on the determinants of gender-gap (Forgasz et al., 2010). Standardized assessment, and in particular PISA studies, have given an increasing importance to investigation on gender differences in academic achievement. Various studies highlight the importance of social factors to explain gender-gap in mathematics, evidencing that in more gender-equal cultures this gap disappears (Guiso et al., 2008). This hypothesis is also supported by the fact that this gap is not present at the early stage of school but it raises during the school years. In this research we endorse this idea and, in particular, we assume that education is the main cause of gender differences. A recent study based on PISA 2009 results confirmed this hypothesis and reveals also that “In addition, gender role attitudes within the family environment […] is found to affect girls’ performance positively.” (González de San Román & De La Rica, 2012).

This research hence fits into a constructivist perspective in which cognitive functions are formed according to the context, and are described as products of social interactions. The learning process cannot be separated from this interactive context defined on the bases of three components: student, teacher and knowledge (Chevallard, 1985). In this paper, we use the idea of didactic contract defined by Guy Brousseau as

> behaviour of the teacher expected from the pupil and the behaviour of the pupil expected from the teacher constitute the didactic contract. (Brousseau, 1980).

The didactic contract imposes rules of behaviour and it is the key to analyse the students response to the items analysed in this paper. The relations established between students and teacher, within the *milieu*, could be also studied in detail through other theoretical constructs, such as the concepts of *coutume didactique* and sociomathematical norms (for a comparison of these concepts and an example of how they can network, see Ferretti et al.,; Ferretti, 2015). Moreover our aim is to study if didactic contract, as a product of social context, have a different influence on male and female and, therefore, on gender gap in maths. At first, it’s interesting to notice that, as we have already seen for gender gap in mathematics, also didactic contract seems to be not present in pre-scholar pupils (Baldisserri et al., 1993) but it origins in primary school.

METHODOLOGY

In this research, we use both quantitative and qualitative analysis. The first part of the analysis is purely quantitative and give us the opportunity to observe the behaviour of the items in a large-scale assessment and to make assumption about that behaviour.

The second part of the analysis has the purpose of validate these assumptions through interviews to a restricted group of students.

Quantitative analysis

The two items investigated in this study belong to 2011 and 2012 INVALSI tests of grade 10. For each test, the data analysed are those of INVALSI's sample. This sample consisted of approximately 40.000 students and it is representative of the population of Italian grade 10 students. The INVALSI team proved the consistency of both tests by using the Classical Test Theory tools and made a first analysis using IRT models and in particular the Rasch Model (INVALSI, 2012b; Rasch, 1960). The Rasch model is a simple logistic model and it is useful to analyse a standardized test such as INVALSI because it allows joint estimation of two kind of parameters: a difficulty parameter for each item and an ability parameter for each student. More specifically, this model express the probability of choosing the correct answer in an item as a function of the item's difficulty and the ability of the students in the whole test and this function is called *Item Characteristic Curve*. In this way, it is possible to use Rasch parameters to represent also the empirical data and, in particular, we can represent the trend of each possible response as a function of the students' ability. Those specific graphs are named *Distractor Plots*.

In the INVALSI National Annual Report 2011, gender differences in math tests are identified on the basis of total medium score observed for male and female (INVALSI, 2011). This gap is perceived in both of the tests analysed and it is statistically significant (INVALSI, 2011; INVALSI, 2012a).

Starting from these INVALSI results and the same dataset, we compare percentages of male and female answers, then we use the Rasch Model to study distractor plots for male and female separately. Distractor plots allow us to study gender differences in relation with the ability level of the students and, in particular, we can observe if there are differences not merely in choosing the right answer but also in trends of the incorrect ones.

Qualitative analysis

The second part of our study is purely qualitative and consists in interviews of a restricted group of students about the items analysed in this paper. For this purpose, we administer in two classes of the same high school a brief questionnaire, consisting in 5 mathematical items including one of the two items studied in this research (Fig. 3). The other items are designed to contextualize the studied task into a mathematical test and to evidence if a student face up the test seriously. Just after the correction of the questionnaire, we select 22 of the 49 students for the interview. We select them on the basis of their response to the task studied and their maths score provided by the teacher. In particular, we choose to interview principally students good at maths (school mark > 6.5/10) who didn't answer correctly to the item. The interviews are semi-structured, task based and in couples. We decide to interview together students that had selected different options and ask them to explain to the classmate the reasons of their decision.

At a later stage, we present them the other item and ask to compare it with the first one. Each interview takes about 20 minutes and is audio taped. At last, we transcribe the interviews and analyse the transcriptions.

ITEMS ANALISYS

In this research, we focus our attention on two similar items: the question intent is the same, both concern the same content (powers properties) and the answers are analogous. Both tasks are multiple-choice questions with only one correct answer but one is set into an algebraic context (Fig. 1) and the other into an arithmetical context (Fig. 3). Moreover, in both items we register a remarkable difference in male and female performances.

The expression $a^{37} + a^{38}$ **is also equal to**

A. ☐ $2a^{75}$

B. ☐ a^{75}

C. ☐ $a^{37}(a+1)$

D. ☐ $a^{37 \cdot 38}$

Figure 1: Item from the grade 10 INVALSI test administered in 2012 [1]

The correct answer is C and it is chosen only by 35% of students. Option A, in which the base is the sum of the basis and the exponent is the sum of exponents, is chosen by 19% of students. Option B and D are similar because the resulting power has the same base of the original ones but the resulting exponent is the sum of two exponent in option B and the product in option D. The 26% of the respondents select answer B, which is the most attractive wrong answer and option D is chosen by 16% of students. In addition, only 3% of students do not respond to this question and this may mean that students are fairly confident about their answers. In the table below, we can also observe that male responded better than female: 38% of male give the right answer compared with 31% of female.

	Total	Male	Female
A	19 %	19 %	20 %
B	26 %	27 %	26 %
C	**35 %**	**38 %**	**31 %**
D	16 %	14 %	19 %
Missing	3 %	3 %	3 %

Table 1: Results of item (Fig. 1) from INVALSI test administered in 2012.

Observing the table (Tab. 1) of percentage, we can also see that the response D is more attractive for girls. These and others particularities of these item responses are more visible using the results of Rasch analysis to graph distractor plots.

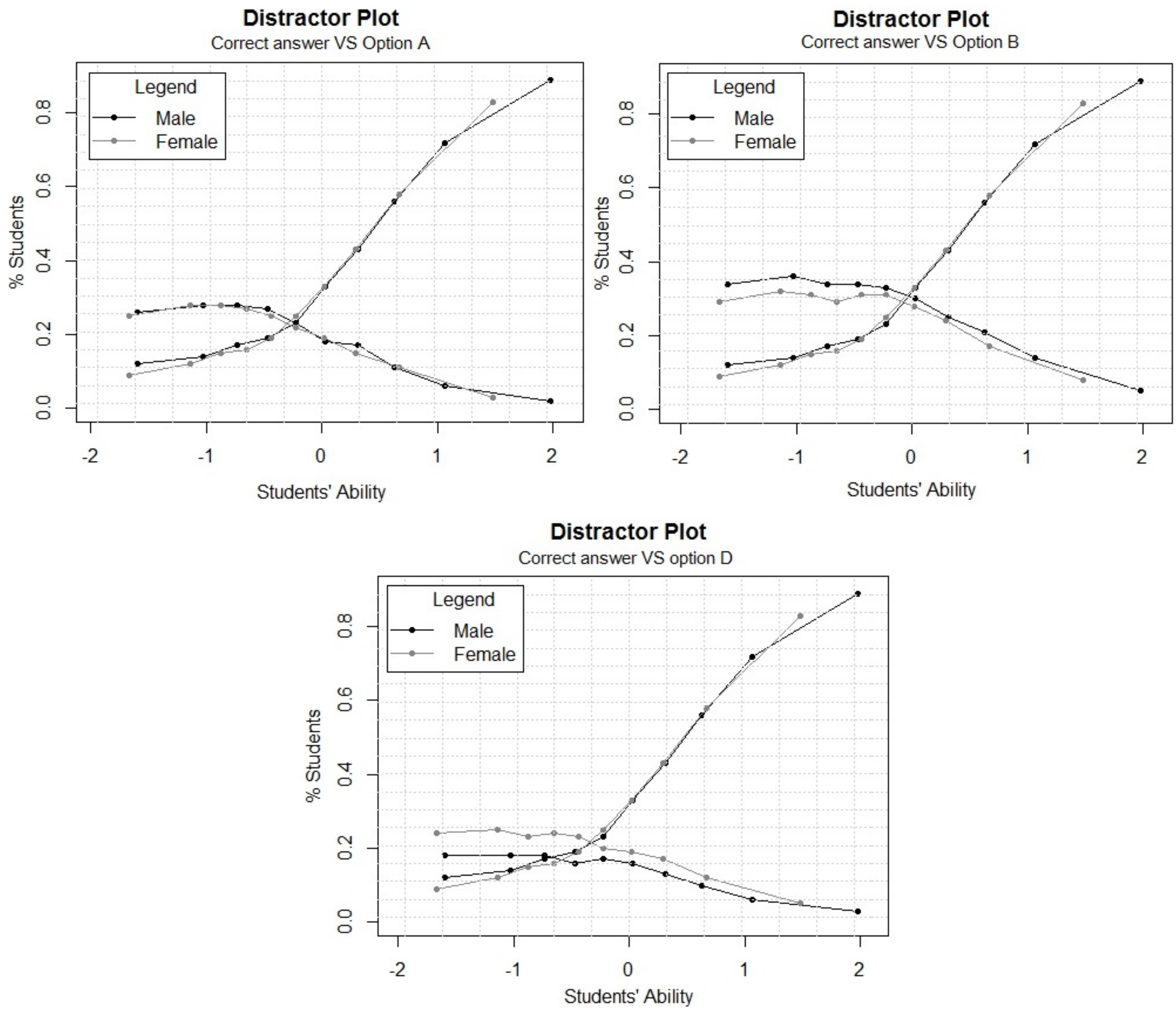

Figure 2: Distractor plots of the first item [1]: comparison between the right answer and the other options (the right answer is the one with increasing trend).

Distractor plots reveal that the trend of the correct answer is almost the same for male and female, although the different percentage seen before. This means that girls and boys with the same ability level choose the correct response with the same percentage, the gender gap observed before in the correct answer (Tab. 1) arises from the fact that female reaching highest ability levels are fewer than male. In Figure 2 we can notice that also the trend of the answer A is almost the same for both male and female but the differences are evident in the behaviour of the other two options. As we observed before, answer D is more attractive for female at all ability levels and, obviously, especially for the lower ones in which it is chosen by a higher percentage of students. Furthermore, boys at all levels of competency prefers the response B compared to girls.

These results become more relevant compared with those of the second item analysed. Indeed, the evidence observed before are all confirmed analysing the second item (Fig. 3) in which the sum of power is analogous but given in an arithmetical context.

The expression $10^{37} + 10^{38}$ is also equal to

☐ A. 20^{75}

☐ B. 10^{75}

☐ C. $11 \cdot 10^{37}$

☐ D. $10^{37 \cdot 38}$

Figure 3: Item from the grade 10 INVALSI test administered in 2011.

This second item results more difficult than the first one and the percentage of correct answers is only 22%. Furthermore, the gender-gap is still present and more relevant than before: only 18% of female choose the correct answer in comparison with the 27% of male. It is very interesting to notice that the trends observed in the distractor plots of the first item, are the same that we can notice in this question, despite the fact that the first one has an higher percentage of correct answers.

STUDENT'S INTERVIEWS

The interviews reveal that most of the students facing with these items are immediately led to identify and apply some kind of rule, in particular those who choose a wrong option. This is a typical didactic contract behaviour. Many students say clearly that, when they see two powers with the same bases they directly think to have to apply powers properties and the reason of this behaviour is inherent into didactical practice:

1	S1:	I had in mind powers rules, I was confused. I didn't even think to option C.
2	I:	So when you see powers you immediately think...
3	S1:	I think to rules. I think: "It will be something with rules".
4	I:	Why?
5	S1:	I don't know, it is what we've always done in the exercises.

Students who choose options B and D remember more or less correctly power properties and for this reason, they expect that the result have the same basis. Then they exclude option A because of the base. Moreover, we observe that options B and D are attractive also for students who know properly powers' properties. In the interview below, for example, the student remember the properties for the product and therefore he excludes option B, but even more he is led to find some other rules to solve the exercise.

1 I: Why students choose $10^{37 \cdot 38}$?

2 S2: Maybe there was a rule according to which to sum powers with the same basis you have to multiply the exponents.

3 I: And this one? Why did they select 10^{75}?

4 S2: Also for this one because there was a rule that says to sum the exponents, but this (rule) is when there is a product! This (answer) is not right because this rule is valid when there is a product!

Students who choose options B and D, often explain their decision on the basis of some kind of rules that derive from their school experience and, in particular, from their relation with the teacher and the milieu habits. Those responses can be related with didactic contract: students choose answers B and D believing that when they solve an exercise that include powers with the same bases they have to apply powers' rules. This behaviour is also observed in students who know powers properties. Moreover, we observed that this attitude belongs to the classroom habits and routine, therefore we can refer this phenomenon to didactic costume seen as the habits picked up in didactical practice during mathematical lessons.

CONCLUSIONS AND FUTURE PERSPECTIVES

This paper presents a study of gender-gap in mathematics from a two-fold point of view: we study the behaviour of male and female facing a mathematical task using specific statistic tool and, in particular, distractor plots. The results of standardized tests analysed using Rasch Model enable us to observe gender differences not only in the whole test, but also focusing on a particular item, objet of our study. Moreover, distractor plots evidence the different behaviour of male and female in choosing each possible answer related their ability in the whole test. The two items analysed in this paper presented the same interesting evidences comparing male and female performances. The gender-gap (in terms of percentage of correct answer) is remarkable in both the tasks. Moreover, distractor plots reveals that, in both the items, Option A have the same trend for male and female, male prefer Option B and female prefer option C at all levels of competency, also for the highest ones. The interviews allow us to interpret these results on the base of students responses and underline that answer B and D can be explained using the construct of the didactic contract. Indeed, the students interviewed always refer to classroom practice, relation with the teacher and the milieu habits. Integrating all these information, we notice that the gender gap, in these particular tasks, is influenced by didactic contract effects: indeed, Option A and the missing percentages are the same for male and female in both the items and therefore the gap in the correct answer percentages, is due to options B and C which are related to didactic contract. We also assume that, in this tasks, male and female are influenced in a different way by didactical contract because, even though both options B and D are related with this construct, the first is preferred by male and the second by female. This different behaviour of male and female could be analyse deeper in future studies, including more interviews.

Finally, the structure of this research could be used also to analyse other items or to study differences not only between male and female but also for other groups of students. Indeed, the Rasch analysis and, in particular, the study of distractor plots shall can provide numerous others evidences that can be interpreted using qualitative analysis and interviews.

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USE OF A MIXED-METHOD DESIGN TO STUDY CREATIVITY DEVELOPMENT THROUGH MODEL-ELICITING ACTIVITIES

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This paper presents one part of a comprehensive study examining the implications and consequences of model-eliciting activities (MEAs) on students' development in several dimensions. This part aims, using a mixed methods research design for analysing qualitative and quantitative data, to explore the effect of an MEA teaching unit on 5th -7th grade students' creative thinking, and to reveal the creative abilities that are involved in the mathematical modeling processes. Findings indicated that engaging students at MEAs improved their creative potential according to the Torrance Tests of Creative Thinking (TTCT) figural test. They also revealed three core categories—appropriateness, 'mathematical resourcefulness' and originality—of students' creative abilities. These findings may give a better understanding of the larger concept of creativity involved in solving mathematical heuristics tasks such as MEAs.

INTRODUCTION

According to the OECD's (2013) report about the challenges facing education, "skills have become the global currency of twenty-first century economies" (p. 11). This report states that countries' competitive advantage depends largely on the development and promotion of certain skills, such as creativity and innovation. The promotion and development of such skills entails in-depth exploration using a mixture of qualitative and quantitative methods, and approach the effectiveness and strength of which have been recently acknowledged by many researchers (Creswell & Plano Clark, 2011).

The development and promotion of students' abilities to solve problems creatively have been explored by many educators and education researchers in a variety of domains, using a variety of different approaches (Torrance, 1974; Chamberlin & Moon, 2005). Model eliciting activities (MEAs) give students opportunities to deal with non-routine, open-ended "real-life" challenges. These authentic problems encourage students to ask questions and be sensitive to the complexity of structured situations while developing, creating and inventing significant mathematical ideas (Lesh & Caylor, 2007; Gilat & Amit, 2014). The implications and consequences of such model problem-solving on the development of students' creative thinking, however, have only been addressed in a few studies (Chamberlin & Moon, 2005). The current paper presents a realistic mixed-method approach to exploring the effect of model-eliciting activities (MEAs) on students' creative thinking. The quantitative inquiry explored the effects of a specially designed modeling teaching unit on the development of students' creative-thinking skills, while the qualitative one explored the mathematical modeling process in order to provide further insight into the creative abilities applied and activated by the students as they engaged in MEAs of "real-life" challenging situations.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 283–290. Szeged, Hungary: PME.

Creativity and Model Eliciting Activities

Guilford (1967) described the creative process as a sequence of thoughts and actions resulting in a novel production, and defined creativity as divergent thinking, consisting of four mental abilities: fluency, flexibility, originality, and elaboration (Torrance. 1974). According to Kruteskii (1976), mathematical creativity appears as flexible mathematical thinking, which involves "switching from one mental operation to another qualitatively different one" (p. 282), and depends on openness to free thinking and exploration of diverse approaches to a problem. Leikin (2009) argued that "solving mathematical problems in multiple ways is closely related to personal mathematical creativity" (p. 133). Other theorists have suggested defining and evaluating creativity based on the apparent outcomes and production results of the creative problem-solving process (Sternberg and Lubart, 2000; Sriraman, 2009). Sriraman (2009) revealed the common characteristics of mathematical creativity through the Gestalt model of the creative process, defining mathematical creativity as the ability to produce a novel or original solution to a non-routine problem. Sternberg and Lubart's (1999) widely accepted definition asserts that creativity is "the ability to produce work that is both novel and appropriate" (p. 3).

Model Eliciting Activities (MEAs) provide the student with opportunities to deal with non-routine "real-life" challenges. These activities are designed according to six principles: reality, model construction, self-evaluation, documentation, sharability and reusability, and an effective prototype (Chamberlin & Moon, 2005; Lesh & Caylor, 2007). This thoughtful design not only engages students in multiple cycles of modeling development, in which they are given the opportunity to construct powerful and creative mathematical ideas relating to complex and structured data (Gilat & Amit, 2014), it also makes it possible to follow students' thinking and reasoning and requires students to represent a general way of thinking instead of a specific solution for a specific context.

Research questions

This paper describes both the design and uses of the mixed-method exploration that sought to answer the following two questions:

1. To what extent, if at all, does experience in MEA workshops develop and improve students' creativity?
2. What are the mathematical creative abilities that are involved in, promoted and encouraged by the modeling process?

Research design

The mixed-method inquiry (Creswell & Plano Clark, 2011) presented in this paper was part of a multilayered exploration aimed at revealing the significant implications of an MEA teaching unit on students' creative and innovative thinking. This unit lasted one academic year and included four workshops based on different MEAs reflecting "real-life" situations, which were worked on by small groups of 3–4 students. Each MEA workshop consisted of three weekly 75-minute meetings and had three parts: a warm-up activity, a modeling activity and a poster-presentation session. The modeling task asked students to solve a mathematically complex “real-life” problem for a hypothetical client. Participants in this research included 157 "high-ability" and mathematically gifted students in the 5th through 7th grades who are members of the "Kidumatica" math club. The “Kidumatica” program provides a framework for the cultivation and promotion of exceptional mathematical abilities in youth from varied socioeconomic and ethnic backgrounds.

<u>The quantitative analysis</u> utilized the standardized TTCT-Figural form (Torrance, 1974) to assess students' creative potential. This test is one of the most commonly used measures of creativity in education and educational research, and has been translated into over 35 languages (Torrance, 2008). The test has two forms: A and B, which were used as pre and post-tests, respectively, and scored according to the Streamline Scoring Procedure (Torrance, 2008). The participants were split into a control group (74 students) and an experimental group (83 students who participated in the teaching unit). Both groups were given TTCT pre- and post-tests before and after the program. The TTCT results were analyzed using repeated measures ANOVA with post-hoc analysis.

<u>The qualitative analysis</u> relied on principles derived from analytical induction (Sriraman, 2009) as well as techniques suggested by Strauss and Corbin (1998) to explore the qualitative data obtained from 12 focal groups (31 students out of 157 participating in the overall research) engaging in different MEA workshops. The data included students' modeling products, video-recordings and classroom observation. Data analysis involved several analytical phases in which data were repeatedly described, interpreted, compared and coded until a coherent interpretation was obtained.

Results and Discussion

<u>The quantitative results</u> revealed significant differences between post- and pre-tests— $F(1, 157) = 47.37$, $p < 0.000$, = 0.23, and between the groups and time (pre-test to post-test)—$F(1, 157) = 28.85$, $p < 0.000$, = 0.15 (Figure 1). This indicates that although both groups started with almost the same creative potential according to the TTCT, after the MEA teaching unit, the experimental group showed greater improvement than the control group.

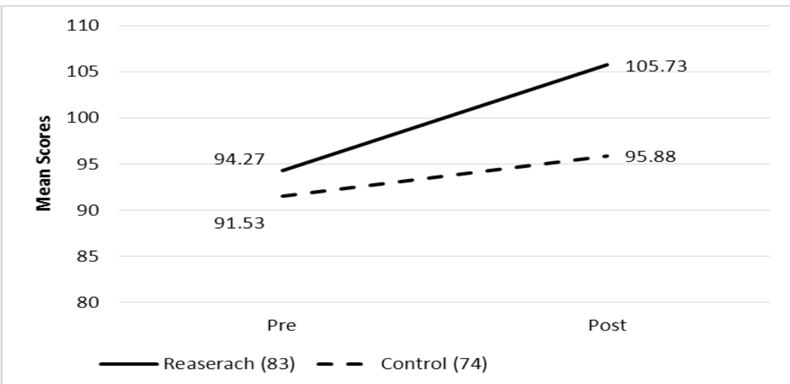


Figure 1: Experiment and control groups' scores on TTCT-Figural pre and post-tests

<u>The qualitative results</u> provided further insight into the mathematical creative-thinking abilities that contributed to and constituted the creative modeling process and its significant outcomes. Three categories and subcategories were formed in light of the theoretical framework and empirical data (see Table 1). Examples illustrating the meaning of the categorization are provided below, using research data from the "Treasure-Chest" MEA of one group of three 7th-grade students (see Figure 2). This MEA was designed specifically for this study, based on a well-known "image-registration" problem that has many applications in different domains. In this modeling task the students were asked to help two boys who had found an old image of their school yard (See Figure 1) with a sign marking the location of a treasure box that is rumoured to have been hidden in the schoolyard decades ago. In order to find the exact location of this treasure box, one of the boys suggested getting an updated image of the school from the Google maps application (See Figure 1). The students were asked to help these two boys to find the exact position of the treasure box and develop a mathematical model that would allow them to transfer the exact position of the treasure from the old image of the school to the updated Google image. They received a ruler and two images: the old image of school with the sign of the treasure and the updated Google image of the school (see Figure 2).

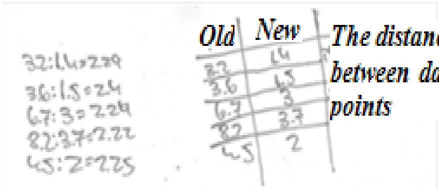
Figure 2: The two images of the school yard ("Treasure-chest" MEA)

The following Ways of Thinking sheets (Chamberlin, 2004) contain the documentation of the students' MEA along with the researcher’s mathematical interpretation of their work (see Figure 3).

WTS – Ways of thinking sheets

Excerpts from students' work

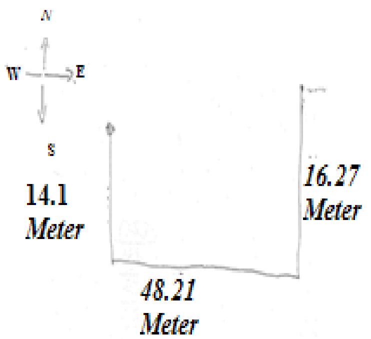
We have found the ratio of width and length between the old map and the new, by taking a few measurements and averaging them using datum points.
We have measured the distance of the way to the treasure (in the old map) and transformed it to the new map, thus arriving exactly at the original place of the treasure
Example of vertical



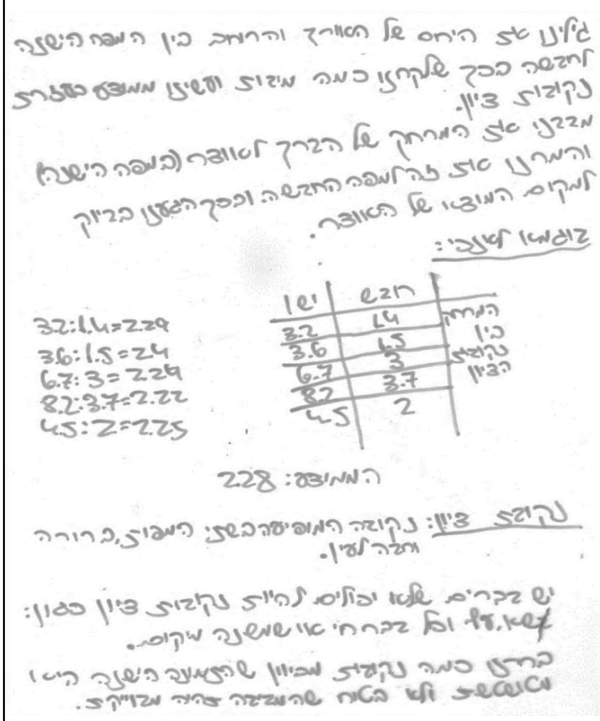
The distance between datum points

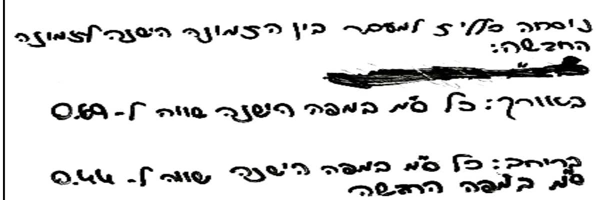
The average [of all ratios]: 228
Datum point: a point that exists in both maps, clear and sharp. There are things that cannot be datum points, such as: grass, trees or any other thing that could change place.
We have chosen several datum points, since the old map is vague and it was not certain that the measurement would be accurate.

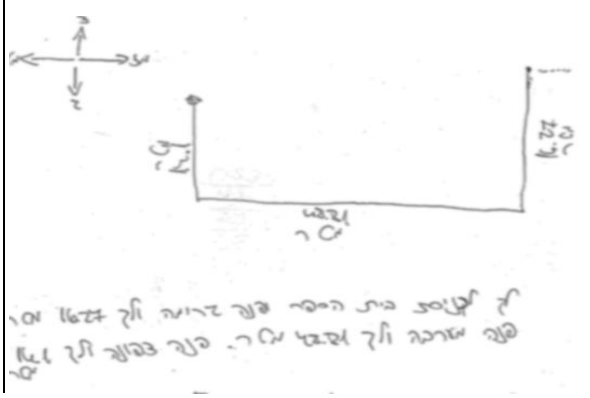
General formulation for the transformation between the old image to the new image:
In the length [horizontal]: every cm at the old map equals-0.69
In the length [vertical]: every cm at the old map equals-0.44



Go to the entrance of the school turn south and walk 16.27 meter. Turn west and walk 48.81 meter. Turn north and walk 14.1 meter.







Students' modeling process description and its formal interpretation

These students went through several development phases, taking measurements between equivalent "stabled" datum points that exist in both the treasure and the google map.

<u>In the first phase</u> they used only vertical measurements, and realized that the vertical identified ratio is different from the horizontal one.

<u>In the second phase</u> they used both horizontal and vertical measurements to calculate the two equivalent ratios.

<u>In the third phase</u> they decided to average the 'ratios' in order to obtain stable transformation.

In the final phase: Their first model used the scale of the treasure map. Then they decided to use two linear transformations that convert vertical/horizontal distances in the treasure map to vertical/horizontal distances in the google map and use the Google scale. According to their verbal explanation during their presentation it "makes more sense", to use a single scale (Google given scale see Figure 2), converting any distance (vertical or horizontal) in the old map to actual distance in their school, and providing an illustration of the path with the exact cardinal directions to the treasure box.

Formal interpretation

<u>**First phase**</u>

$$R_{vertical} = \frac{Vertical\ distance\ (treasure\ map)}{Vertical\ distance\ (google\ map)}$$

<u>**Second phase**</u>

$$R_{horizontal} = \frac{Horizontal\ distance\ (treasure\ map)}{Horizontal\ distance\ (google\ map)}$$

$$R_{vertical} = \frac{Vertical\ distance\ (treasure\ map)}{Vertical\ distance\ (google\ map)}$$

<u>**Third phase**</u>

$$R_{horizontal} \frac{= \sum(Horizontal\ distance\ (treasure\ map))/(Horizontal\ distance\ (google\ map)}{n}$$

$$R_{Vertical} \frac{= \sum(Vertical\ distance\ (treasure\ map))/(vertical\ distance\ (google\ map)}{n}$$

<u>**Final phase I**</u>

$$Dy_{vertical} = y\ /\ S_{treasure\ map(vertical)} \qquad Dx_{horizontal} = x\ /\ S_{treasure\ map(horizontal)}$$

$$(scale)S_{treasure\ map\ (vertical)} = \frac{S_{google\ map}}{R_{vertical}} \qquad S_{treasure\ map\ (horizontal)} = \frac{S_{google\ map}}{R_{horizontal}}$$

<u>**Final phase II**</u>

$$Dy_{vertical} = y * \frac{R_{vertical}}{S_{google\ map}} \qquad Dx_{horizontal} = x * \frac{R_{horizontal}}{S_{google\ map}}$$

$$Dy_{vertical} = desired\ distance \qquad Dx_{horizontal} = desired\ distance$$

$$y = vertical\ distance\ at\ the\ treasure\ map \qquad x = horizontal\ distance\ at\ the\ treasure\ map$$

Figure 3: WTS documenting 7th-grade students' "two-dimensional" model

Core & Sub-categories		Description	Example illustrating the meaning of the categorization
1. Appropriateness	Utility	Deliberate actions or means applied by students to generate useful solutions, not only for the current situation, but for other similar situations as well (reusable).	In their oral presentation students explain how they deliberately developed a useful, effective and easy to use conceptual mathematical tool that can convert the (transformed) distances into "reality" using a single scale the given Google scale.
	Documentation	Students' ability to apply varied representations to present and share information with others (sharable).	Excerpts from students' work in Figure 3 show how students used tables, drawings and written explanations to mathematically communicate "how" they were actively attempting to make sense of the structured problematic "real-life" situation in a way that could be sharable with others.
2. Mathematical Resourcefulness	Fluency	Students' tendency to consider or evaluate several ideas and perspectives.	The early stages of the students' modeling process, which involved the fluent generation of different relevant mathematical objects, such as different types of datum points and the four cardinal directions, before an effective solution emerged.
	Flexibility	Students' ease in switching from one mental operation to another, applying redefinition and transformation, and finding new ways to describe both the data set and its behavior.	The different development phases in Figure 3 reflect students' ease in switching from one mental operation to another to verify their early conceptualization of the situation and "discover" further information and more relationships within the data to better describe their advanced interpretation,
	Elaboration	Students' refinement, generalization and integration abilities, as applied to developing a new level of more abstract or formal understanding.	Students' final phase, demonstrated in Figure 3, shows how students elaborated (extended, refined and integrated) their ideas to develop a new level of more abstract or formal understanding and create a more generalized conceptual tool that involves 'only' one scale, which according to their verbal explanation during their presentation *"makes more sense"* than the use of two scales.
3. Originality		Students' ability to break away from routine or bounded thinking to create unique and powerful mathematical ideas that differ from those developed by most other students.	Although there were other groups that found a linear transformation that converts vertical and horizontal measurements/distances from the treasure image to the Google image based on the ratio between equivalent distances, only this group produced a *"general formulation for the transformation from the old image to the new image"* that allows the use of a single scale (Google given scale) for conversion to "real-life" actual distance.

Table 1: Description and examples of each establish category and its sub-categories

Final remarks

This paper highlights the significant advantages of using both qualitative and quantitative approaches (Creswell & Plano Clark, 2011) to explore the development of students' creative abilities through MEAs. The quantitative results revealed the positive effect of the MEA teaching unit on the promotion of students' creative abilities, supporting other educational studies and explorations (Torrance, 1974; Levav-Waynberg & Leikin, 2012) that showed that significant training in creative problem solving can develop students' creativity. The following qualitative examination provided us with deeper insights into what is involved in the creative mathematical process of young students engaging in non-routine, “real-life”, structured problem-solving (Sriraman, 2009) through MEAs. These results underline the significant role of mixed-method design in educational studies aimed at exploring multifaceted and complex phenomena, such as the development of students' creative abilities, following Creswell & Plano Clark (2011) who argued that "the combination of quantitative and qualitative data provides a more complete understanding of the research problem than either approach by itself" (p.8).

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ONTO-SEMIOTIC CONFIGURATIONS UNDERLYING DIAGRAMMATIC REASONING

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Diagrams and in general the use of visualization and manipulative material play an important role in mathematics teaching and learning processes. Although several authors warn that mathematics objects should be distinguished from their possible material representations, the relations between these objects are still conflictive. In this paper, some theoretical tools from the onto-semiotic approach of mathematics knowledge are applied to analyse the diversity of objects and processes involved in mathematics activity, which is carried out using diagrammatic representations. This enables us to appreciate the synergic relations between ostensive and non-ostensive objects overlapping in mathematical practices. The onto-semiotic analysis is contextualised in a visual proof of the Pythagorean theorem.

INTRODUCTION

The use of different representations, visualizations, diagrams, manipulative materials, are proposed to favour mathematics learning by assuming that such materials make up representations of mathematics concepts and of the structures in which they are organised. It is supposed that the use of material representations is necessary, not only to communicate mathematical ideas but also for their own construction. However, the relations between representations, objects and construction of meanings are still conflictive. This issue is key for mathematics education since "any didactic theory, at one moment or another (unless it voluntarily wants to confine itself to a kind of naïve position), must clarify its ontological and epistemological position" (Radford, 2008, p. 221).

Researches in diagrammatic reasoning and about the use of visualizations in mathematics education do not usually deal with the type and diversity of mathematical objects. In this paper, this problem is faced using some theoretical tools from the onto-semiotic approach (OSA) (Godino, Batanero, & Font, 2007; Font, Godino, & Gallardo, 2013). Mathematical objects are considered to be abstracts whereas diagrams are specific and perceptible. It is necessary not confuse them, but the relationship between both types of objects are not dealt with explicitly. This situation is not strange since to clarify what abstract objects are, and their relationship with the empirical world, is a full-scale philosophical and psychological problem, which is addressed from different paradigms and theoretical frameworks.

In the OSA it is assumed that mathematics is a human activity (anthropological postulate) and that the entities involved in this activity come or emerge from the actions

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 291–298. Szeged, Hungary: PME.

and discourse through which they are expressed and communicated (semiotic postulate). The epistemological, semiotic, and educational problem that interests us is to clarify the relationship between the visual, diagrammatic or iconic representations, and the non-ostensive mathematical objects that necessarily are involved.

In the following section, some characteristic features of the diagrammatic reasoning that point out the problem mentioned are described, that is the gap between the representation and the mathematical object represented. Then, the notion of ontosemiotic configuration of practices, objects and processes is summarised. This theoretical tool is used to analyse the diagrammatic reasoning in a visual proof of the Pythagorean theorem. In the final section, some reflections about the type of understanding that the onto-semiotic approach to mathematical knowledge might provide to diagrammatic reasoning are included.

DIAGRAMMATIC REASONING

In mathematics education, talking of diagrammatic reasoning means entering into the field of Peircean Semiotics (Dörfler, 2005; Bakker & Hoffmann, 2005; Rivera, 2011), although the use of diagrams as a resource of thought and scientific work is also found in other fields and disciplines (Shin & Lemon, 2008).

A double conception about the notion of diagram is found: one wider conception, in which any type of inscription that makes use of the spatial positioning in two or three dimensions (right, left, forward, backward, etc.) is a diagram (geometric figures, graphs, conceptual, etc.). Another more restricted conception requires being able to carry out specific transformations, combinations or constructions with these representations, according to certain specific syntactic and semantic rules. In this research report, it is justified why this second approach should be retained.

Diagrammatic reasoning involves three steps (Bakker & Hoffmann, 2005, p. 340): the first step is to construct a diagram (or diagrams) by means of a representational system; the second step is to experiment with the diagram (or diagrams); the third step is to observe the results of experimenting and reflecting on them.

Duval (2006) attributes an essential role not only to the use of different systems of semiotic representation (SSR) for mathematics work but also to the treatment of the signs within each system and the conversion between different SSR:

> The role that signs play in mathematics is not to be substituted for objects but for other signs! What matters is not representations but their transformation. Unlike the other areas of scientific knowledge, signs and semiotic representation transformation are at the heart of mathematical activity. (Duval, 2006, p. 107)

Dörfler (2005) recognises that diagrams can make up a register of autonomous representation to represent and produce mathematics knowledge in certain specific fields; however, it is not complete. It requires to be complemented by conceptual-verbal language in order to express notions like: continuity and differentiability;

impossibility that specific objects exist; using the quantifiers ‘for all’, ‘each one’ and ‘there are’.

> For our purposes here, it is very important to make a clear distinction between "diagrams" and all kinds of representations, visualizations, drawings, graphs, sketches, and illustrations as widely used in professional mathematics and in mathematics education as well. Although these might be diagrams in the specific sense used here, this is mostly not the case. This is due to the lack of the constituting operations by which an inscription or visualization becomes only a diagram. (Dörfler, 2005, p. 58)

Shin & Lemon point out another problem related to the use of diagrams:

> A central issue, if not the central issue, was the generality problem. The diagram that appears with a Euclidean proof provides a single instantiation of the type of geometric configurations the proof is about. Yet properties seen to hold in the diagram are taken to hold of all the configurations of the given type. What justifies this jump from the particular to the general? (2008, section 4.1)

Sherry (2009) adopts an anthropological perspective on the role of diagrams in mathematics argumentation, which involves an objectification of the empirical reality. This perspective differs from the Peircean semiotic, according to which diagrams are an essential means in the process of hypostatic abstraction. Sherry analyses the role of diagrams in mathematics reasoning (geometric and numerical – algebraic) without resorting to the introduction of abstract objects and relying on a Wittgensteinian perspective of mathematics. “Recognizing that a diagram is just one among other physical objects is the crucial step in understanding the role of diagrams in mathematical argument” (Sherry, 2009, p. 65).

In this position, the author avoids recurring to abstract conceptions which are conceived in an empirical-realistic way (hypostatic abstraction) in order to understand them as socially agreed grammatical rules, about the use of languages through which we describe our worlds (material or immaterial).

> I have emphasized that diagrammatic reasoning recapitulates habits of applied mathematical reasoning. On this view, diagrams are not representations of abstract objects, but simply physical objects, which are sometimes used to represent other physical objects. (Sherry, 2009, p. 67)

ONTO-SEMIOTIC CONFIGURATIONS

In the OSA framework, it is proposed that six types of objects intervene in mathematics practice, which can be contemplated from five dual points of view (figure 1) (Font et al., 2013). The non-ostensive (immaterial) entities: conceptual, propositional and procedural, are conceived as rules. The Wittgenstein’s anthropological view is assumed, according to which concepts, propositions and mathematics procedures are empirical propositions, which have been socially reified as rules. Sherry clearly and synthetically describes this Wittgensteinian conception of mathematical objects:

> In order for an empirical proposition is harden into a rule, there must be overwhelming agreement among people, not only in their observations, but also in their reactions to

> them. This agreement reflects, presumably, biological and anthropological facts about human beings. An empirical proposition that has hardened into a rule very likely has practical value, underwriting inferences in commerce, architecture, etc. (Sherry, 2009, p 66)

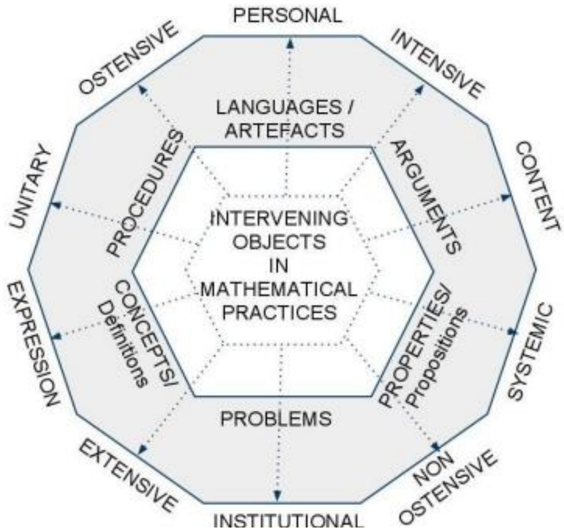

Figure 1: Objects that intervene in mathematical practices (Font et al., 2013, p. 117)

Both the dualities and the configurations of primary objects may be analyzed from the process/product perspective. The objects of a configuration (problems, definitions, propositions, procedures and arguments) emerge through the respective mathematical processes of communication, problematization, definition, enunciation, development of procedures (algorithms, routines, etc.) and argumentation. For their part, the dualities give rise to the following cognitive/epistemic processes: institutionalization-personalization; generalization-particularization; analysis / decomposition - synthesis / reification; materialization / concretion - idealization / abstraction; expression / representation - signification.

Behind diagrammatic reasoning, and the use of manipulative teaching materials, there is an implicit adoption of an empirical – realistic position about the nature of mathematics. This position does not recognize the essential role of language and the social interaction in the emergence of mathematical objects. To a certain extent, it is supposed that the mathematical object “is seen”, it is hypostatically detached from empirical qualities of things collections. Against this position, the anthropological conception of mathematics proposes that concepts and mathematical propositions should be understood, not as hypostatic abstractions of perceptual quality, but as regulations of the operative and discursive practices carried out by people in order to describe and act in the social and empirical world in which we live.

This anthropological way of understanding abstraction, that is, the emergence of general and immaterial objects forming mathematical structures, has important consequences for mathematics education since mathematics learning should take place through students’ progressive participation in the mathematics language games. For example, in the current introduction of dynamic software in school is necessary to evolve their use according moments of exploration, illustration and demonstration (Lasa & Wilhelmi, 2013), which allow an understanding, reuse and construction of new mathematical knowledge. In this way, dialogue and social interaction take on an important role, in comparison with the mere manipulation and visualization of ostensive objects.

ONTO-SEMIOTIC CONFIGURATION IN A VISUAL TASK

In this section, the types of practices, objects and processes put at stake in the statement and demonstration of the Pythagorean theorem are analysed. Usually it is presented as a visual or "without words" demonstration. It is shown that, indeed: “picture-proofs don’t show their results on their sleeve, as it were; it’s necessary to study them for a while, before they reveal their treasure” (Sherry, 2009, p. 68).

Task

What is the relationship between the areas of the figures shaded A and B?

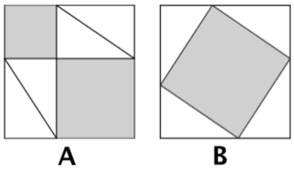

Figure 2: A visual proof of the Pythagorean theorem

The following sequence of operative and discursive practices is one possible answer[5]:

1. We assume that the representations in Figure 2 are squares and right triangle, and the lengths of their sides are indeterminate: *a, b, c* (Figure 3).
2. The quadrilaterals formed by the outer segments of the figures A and B are congruent squares because the sides have equal length, ($a + b$).

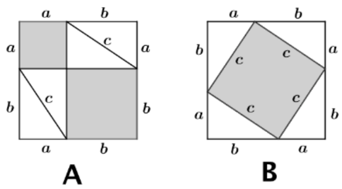

Figure 3: Metrics hypothesis needed

3. The representations of right triangles in A and B are congruent because their sides are of equal length.
4. The shaded region in Figure A is equal to the shaded region in Figure B. This is because two squares of equal area are formed of four equal triangles.

[5] Explanatory proof (Cellucci, 2008).

5. The shaded area in Figure A is the sum of the squares area of sides a and b, respectively, $a^2 + b^2$.
6. The shaded area in Figure B is the square's area of side c, c^2.
7. The shaded regions are interpreted as areas of the squares whose sides are the legs and hypotenuse of the triangle, respectively (Figure 4).

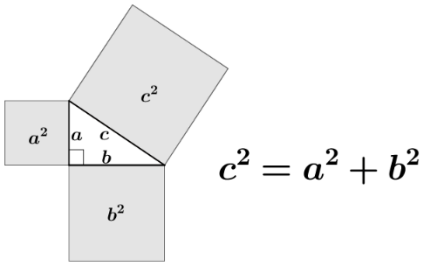

Figure 4: Determination of the Pythagorean theorem

8. Then, the area of the square on the hypotenuse is equal to the sum of the squares areas on the other two sides: $c^2 = a^2 + b^2$.

Configuration of objects and meanings

In the first column of the Table 1, the expressions in ordinary language (sequential) is summarised; such expressions are added to the diagrams to produce the justification and explanation necessary of the theorem. In the second column, the system of 'non-ostensive objects' is included. In addition, how the 'ostensive / non-ostensive' duality, and the "example / type" (particular / general) duality are linked to the intervention of concepts, propositions, procedures, and arguments are shown.

Our analysis agrees with and supports Sherry´s position about the use of diagrams in mathematics work: rather than building an accurate diagram, what matters is the mathematical knowledge involved, which is not visible anywhere; it is not in the diagrams themselves. In the case of using dynamic software, it is essential to progress from moments of illustration (where objects can be manipulated with great precision) to moments of demonstration (where objects are not essential, rather the construction process of diagrams). This way, features of specific *examples* can progress towards the corresponding structural *type*. In general, the diagram supports or makes possible the necessary process of particularization of the general rule; it makes the conceptual object intervene in order to participate in a practice from which another new conceptual object will emerge (in our example, Pythagorean theorem).

OSTENSIVE OBJECTS (Means of expressions)	NON - OSTENSIVE OBJECTS (Concepts, propositions, procedures, arguments)
Task statement: *What is the relationship between the areas of the figures shaded A and B?* (Figure 2)	*Concepts*: area (extension of a plane region), sum of areas; comparison of areas. *Particularization*: these concepts are particularized to the case of the figures given. The squares, triangles and the relationships between the areas, are generic.
1. We assume that the representations in Figure 2 are squares and right triangle, and the lengths of their sides are indeterminate: *a, b, c* (Figure 3).	*Concepts*: square, right triangle, side, indeterminate measurement of length. *Particularization*: these concepts are particularized to the case of the figures given. The figures refer to square and triangle generics. The lengths are generic.
2. The quadrilaterals formed by the outer segments of the figures A and B are congruent squares because the sides are of equal length, ($a + b$).	*Proposition*: the two exterior squares are congruent. *Argumentation*: because the sides of the squares have the same length. This is ($a+b$). The proposition is general; it is valid for the "examples" (figures) and for any "type". This is an essential hypothesis in the explanatory process.
...	
8. Then, the square's area of the hypotenuse is equal to the sum of the squares areas of the other two sides: $c^2 = a^2 + b^2$.	*Proposition*: thesis (Pythagorean theorem) *Justification*: steps 1 to 7. It is geometrically interpreted (comparison of areas). It is also interpreted in arithmetic / algebraic terms (numerical relationships).

Table 1: Configuration of objects and meanings

FINAL CONSIDERATIONS

The function that we attribute to the diagrams helps to surpass ingenuous empiricist positions about the use of manipulatives and visualizations in the processes of mathematics teaching and learning: there is always a cohort of intervening non material objects which are essential to solve these situations accompanying the necessary materializations that intervene in the situations-problems and the corresponding mathematics practices. However, this layer of material objects should not prevent seeing the layer of immaterial objects that really make up the conceptual system of institutional mathematics. Both layers are interwoven and to a certain extent are inseparable. Mathematics teacher should have knowledge, understanding and competence in order to discriminate the different types of objects that intervene in school mathematics practice, based on the use of different systems of representations and being aware of the synergic relations between the same.

Acknowledgment

This report has been carried out in the frame of the research Projects EDU2012-31869 and EDU2013- 41141-P, Ministerio de Economía y Competitividad (MINECO) Spain.

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SCHOOL MATHEMATICS KNOWLEDGE AND STUDENTS' MATHEMATICAL ACTIVITY

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We report on an investigation of how school mathematics knowledge is deployed differently by a group of students in their mathematical activity in autonomous peer work and in the communication of this work to the teacher. We found 'moves' between certain epistemic issues arising in peer work and some of these issues not being communicated in the subsequent interaction with the teacher. The metaphor of a move serves us to focus the analysis on two moments of the students' mathematical activity in order to identify potential differences in the use of knowledge concerning the construction and justification of school mathematics knowledge. To illustrate this type of move we discuss two turns of the interaction from the perspective of the related epistemic issues that are either visible or invisible when analyzing them.

INTRODUCTION

We discuss results from the identification and explanation of epistemic issues involved in and concerning students' mathematical activity in two contexts of a secondary mathematics classroom: autonomous peer work and group interaction with the teacher. These are social contexts of mathematical activity for which similar practices may be recognized and valued differently depending on who is involved in their discussion at each moment (Goizueta, 2015). We argue that a clear case needs to be made about the criticality of certain moves in the students' mathematical activity, between certain epistemic issues arising in peer work and some of these issues not being visible in the interaction with the teacher. The existence of such moves suggests a lack of systematic inquiry and reflection about epistemic issues in the researched setting.

Epistemic issues in school mathematics have often remained undetected, and are sometimes thought of as undetectable, in the analyses of mathematical activity in the mathematics classroom. In order to examine how students become involved in and refer to epistemic issues in particular contexts of interaction in the mathematics classroom, we empirically place our study in compulsory secondary mathematics education. We interpret epistemic issues as related to the practices of school mathematics whose use and learning prepare students of these ages to construct, reflect on and justify mathematical knowledge in a range of situations. In this way, our definition of *epistemic* relates to knowledge about the construction and justification of school mathematics knowledge, which is traceable across different moments of the students' mathematical activity in the classroom. Epistemic issues are not always evident in the students' explicit discussion but may be found in the interpretations made by the researchers of what is inferable from what has been said and done.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 299–306. Szeged, Hungary: PME.

THE MATHEMATICAL CULTURE OF THE MATHEMATICS CLASSROOM

Like Steinbring (2005), we consider social, cultural and historical processes as constitutive of mathematical activity, and such activity in turn as related to kinds of participation in particular mathematical cultures. Thus, mathematics shows specific features in different contexts of development and practice, relative to the persons and groups participating in its production and their circumstances. The context in our study is the mathematics classroom, with a mathematical culture configured, on a macro level, by the meanings historically attributed to school mathematics (Radford, 2008) and, on a micro level, by the local interactions between the students and the teacher (Krummheuer, 2011). The double micro and macro configuration of the mathematical culture of the mathematics classroom points to complex processes of meaning construction taking place in the interaction. Micro-social meanings (i.e. situated meanings enacted by individuals in their interaction with others) and macro-historical meanings (i.e. wider meanings enacted by and on others) dialectically evolve in a reflexive relationship that makes it difficult to distinguish them during mathematics teaching and learning, and with respect to their impact on the continuous (re)creation of school mathematics knowledge.

The adoption of a micro-macro theoretical perspective situates the mathematical culture of a mathematics classroom as an emerging result of the interaction among participants as representatives of macro-historical meanings, rather than as some pre-given object deployed by the teacher as the representative of the mathematics discipline. Participation in this mathematical culture is therefore not reducible to interpreting and producing some propositional mathematical knowledge (e.g. definitions, theorems) directly linkable to the mathematics discipline. It also and fundamentally requires understanding about why and how to justify the use of this knowledge in the classroom, as well as when to do so and with whom at each moment. In this respect, the mathematical culture of any mathematics classroom is framed by and within uses of school mathematics knowledge and related meta-claims about the conditions of construction and justification of instances of this knowledge.

An important part of the knowledge involved in the identity practices of the mathematical culture of the mathematics classroom is, therefore, knowledge about the construction and justification of school mathematics knowledge. Ernest (1998) distinguishes between knowing-that, which refers to propositional knowledge (e.g. the Pythagorean Theorem), and knowing-how, which “refers to practical knowledge, skills, or dispositions, which are not immediately given in the form of propositions, even if they can (or cannot) ultimately be so represented” (p. 136). Knowing direct proportionality in (school) mathematics, for example, is not only to be able to state propositions such as ‘when two variables are directly proportional their ratio is constant’; one has to be able to identify when is it appropriate to work with them, and to reflexively justify the adequacy of certain adapted uses.

Some of this mathematical knowing-how constitutes tacit knowledge that "is validated by public performance and demonstration" (Ernest, 1999, p. 67), and is largely implicit in the mathematical culture of the mathematics classroom. For example, the knowledge necessary to use a particular semiotic representation in certain cases of mathematical practice (e.g. the tree representation of probabilistic phenomena) will usually be tacitly produced and conveyed by its application in paradigmatic cases, its repetitive application to recognizable types of cases, and its adaptation in tackling novel cases. Through its use in public practices, this tacit knowledge becomes socially justified in the mathematics classroom, and can thus be properly termed knowledge. On the other hand, some of this tacit knowledge is epistemological in nature, in the sense that it relates to how school mathematics knowledge is constructed, reflected on and justified. What we then have is that the mathematical culture of the mathematics classroom is constructed by the enactment of micro-social and macro-historical meanings which are either explicitly or tacitly introduced and validated through a network of knowing-that and knowing-how practices.

THE EXPERIMENT AND THE RESEARCH QUESTIONS

A major concern for the empirical design of the study was to determine what kind of classroom dynamics, mathematical task and curricular content could facilitate the emergence of epistemic issues in the students' activity during two one-hour lessons. In collaboration with the teacher, we planned a problem-solving scenario with time for small group work and whole class discussion. It was decided that an everyday-context problem could trigger reflection on and justification of mathematical work through a wide range of arguments. Moreover, we were interested in curricular contents that were novel for the students, since this would be likely to prompt more flexible uses of knowledge and less reproductive approaches to school mathematics. Contents regarding early concepts of probability, chance and randomness were finally chosen. We proposed to the teacher a modified variation of a coin game problem used by Paola (1998) in a classroom experiment, which is itself a variation of a historical problem discussed by Pascal and Fermat in 1654. Our problem was posed as follows:

> Two players are flipping a coin in such a way that the first one wins a point with every head and the other wins a point with every tail. Each is betting €3 and they agree that the first to reach 8 points gets the €6. Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5. How should they split the bet? Justify your answer.

In Goizueta, Mariotti and Planas (2014), we used the same problem and lessons with thirty 14/15-year-old students in Barcelona, Catalonia-Spain, to examine the processes of resolution by a group of students, and specifically the validation of a proposed mathematical model. On that occasion, it was seen that work on this problem promotes: i) the use of arithmetical knowledge and ii) the elaboration and combination of deductive, inductive and abductive types of arguments. We asked the teacher to avoid showing either approval or disapproval of the students' numerical answers and strategies. Instead of hint-guiding the students and valuing traces of probability reasoning, we wanted her to foster the discussion of competing (proportional, statistical, probabilistic...) models in the approaches to the problem.

Two groups of students (EA and EB) were video-taped during the two lessons and their written resolution protocols were taken for complementary revision of what was carried out in the group. Furthermore, two video-taped interviews, one for each group, were conducted by the first author one week after the second lesson. The interviews were planned to guide, revise and complete our preliminary interpretations of what happened in the lessons. We watched the videos of the two groups several times, in order to search for moments of the students' mathematical activity in which epistemic issues regarding school mathematics knowledge were detectable in the discussion, during the resolution of the problem. We used some video-clips of these moments in the interviews as a way, on the one hand, to make direct references to the lessons and, on the other, to have common and contrastable data for discussion with the students.

The research of epistemic issues involved in and concerning the students' mathematical activity followed a qualitative approach. It was guided by two questions:

1) Do the students become involved with or refer to knowledge about the construction or justification of school mathematics knowledge?
2) What is the evidence for this concerning the interpretation, understanding and resolution of the proposed problem?

Inductive and iterative analyses were developed concurrently through methods of constant comparison. The triangulation with researchers in the team helped to infer epistemic issues from transcripts and video-clips in which some of the micro-social and macro-historical meanings attributed to the students' mathematical activity were not overtly explicit. We created an initial list of epistemic issues which were examined for relationships and overlaps and then refined into a shorter list. At an advanced stage, all the epistemic issues, whose presence had been inferred from the analysis of lesson accounts of the mathematical activity in EA and EB, were further examined with a focus on whether and how these same issues could be detected in the communication of group work to the teacher. It was throughout this process that two more questions, which cannot be answered independently of the first ones, came up:

3) Do the students become involved with or refer to knowledge about the construction or justification of school mathematics knowledge differently in their communication with the teacher?
4) What is the evidence for this difference concerning the interpretation, understanding and resolution of the proposed problem?

The third and fourth questions came after the research in Goizueta et al. (2014) and Goizueta (2015). It took time to begin to understand the fact that the students might be enacting certain instances of school mathematics knowledge when the teacher was involved in the interaction with them, and others when the group was working autonomously. In both Goizueta et al. (2014) and Goizueta (2015), there was a search for and detection of epistemic issues under a similar approach. Nevertheless, in these studies our results were related to classroom episodes of the students' mathematical activity and specific criteria for comparison between episodes were not considered. At that time, the focus was on the particular features of the mathematical activity in each episode, and not so much on the variability of these features across episodes.

EPISTEMIC ISSUES IN THE STUDENTS' MATHEMATICAL ACTIVITY

We broadly take the metaphor of a move to indicate the relationship between what is done and said by the students in peer work to solve a problem and what is subsequently communicated to the teacher. In these two contexts of classroom interaction, we research epistemic issues regarding how the students refer, either explicitly or implicitly, to knowledge about the construction and justification of school mathematics knowledge. Thus, the metaphor of a move helps to focus the analysis on two moments of the students' mathematical activity and identify differences in terms of the epistemic issues involved. Below we illustrate a particular move that goes in the direction of the invisibilization of epistemic issues that were deployed by the students in peer work. This is a type of move that has been found for EA and EB in the two lessons. If we see peer work and group discussion with the teacher as two positions, in these positions the students become differently involved in or concerned with certain epistemic issues during the problem resolution.

To illustrate our argument, in this report we use lesson data from the two classroom contexts and one of the groups, EA. These data reveal part of the phenomenon of: i) some epistemic issues being addressed by the students during peer work; and ii) some of these epistemic issues not being visible –and perhaps being avoided– in the communication with the teacher. We choose two turns of talking that were discussed in the interview with the students together with other turns, and for which the inferred tacit knowledge was contrasted with them. Although we do not provide empirical evidence of the role of the exemplified turns in the interaction, our analysis indicates that in these turns the discussion of concrete models of reasoning was initiated.

Visibility of epistemic issues in peer work

In an initial stage of the mathematical activity of EA, Anna, one of the students in this group, shared the following thought with her three peers:

> Anna: This one only needs to get one point and this one three to get to six euros. But obviously, because it's random, the game, you know, one's got more chances. Because imagine that now, suddenly, if the game didn't stop... you could get three tails in a row and then this one would win. So A does have more chances of winning but B could win as well.

Anna refers to the proposed situation in the wording of the problem as a random game. By resorting to randomness, she tacitly brings to the conversation references that are rooted in prior experiences with coin tossing and random games, as well as public discourses about them (e.g. it is equally likely to obtain heads or tails when tossing a coin). These references outline certain aspects of the situation in relation to the random nature of the game. In this way, the student suggests an interpretive framework to make sense of the situation. Anna draws on this framework to account for the empirical dimension of the problem: having won more points amounts to having an advantage of some sort ("So A does have more chances of winning (…)"). Here, a scenario in which B wins the game ("(…) but B could win as well") is foreseen as feasible, though the special case of B obtaining "three tails in a row" is less expected.

In this turn we can distinguish three related issues that come into focus and which, together, are of epistemic significance: i) shared references with respect to the random nature of the game; ii) a particular account of the situation; and iii) the knowledge involved in the use of such references to construct this account. The epistemic significance of these interrelated issues can be elucidated by exploring the turn as an expression of tacit taken-as-shared knowledge regarding adequate ways of reasoning and proceeding ("(...) because it's random, the game, you know, one's got more chances"). The words "because" and "so" are marks indicating a causal relationship between the random nature of the game and the particular account of the situation. This is actually an interesting example of how the assumption of certain shared references helps to sustain a reasoning that may not be mathematically well founded, since these students have not been taught probability theory at school.

We examine this turn by Anna because it acts as a catalyst that precipitates other considerations regarding "good reasons" to be used in situations of randomness. In the interaction within EA and EB, there are a few turns that are especially 'strong' from the perspective of, on the one hand, the epistemic issues that become visible when analyzing them in isolation and, on the other, their influence on the development of epistemic contents of other turns. These 'strong' turns and their related analyses are sufficiently significant to conclude that epistemic issues concerning school mathematics knowledge were explicitly present in peer work.

Invisibility of epistemic issues in the interaction with the teacher

One of the 'strong' turns of the interaction with the teacher is again a particular turn by Anna when this student first reports the mathematical activity of her group:

> Anna: We thought that... well, player A has got seven points and B five points. We thought that if they won four points each, three euros for each one, and the distribution would be fair. Then we did six euros divided by eight, which is the total, by how many points... You know? I mean, how much one point would be, eight points in total. But we said, no, no, no. They have twelve points in total… and we multiplied each point by zero point five (...).

When addressing the teacher, Anna starts by elaborating on a hypothetical tied game ("(...) if they won four points each, three euros for each one (...)"). She thus proposes a generic example: in case of a tie, no matter what the score is, the money should be distributed equally. There is, however, an absence of elements to justify what is said. This might indicate that Anna expects the teacher to share common references about the situation and about the construction of the generic example, sufficient for it not to be controversial or in need of further explanation. Also, Anna qualifies this distribution as “fair”, using a term that was mentioned at the beginning of the lesson by the teacher, when she presented the problem (“You have to explain what you consider to be a fair way to distribute the money”). Although Anna does not explain her use of “fair” in this turn, it is seemingly an indication of the solution’s validity. We cannot know, though, whether she suggests the notion of fairness to describe the result in semantically significant terms or whether she is matching the teacher’s vocabulary to recognizably convey a sense of validity. Anna then advances one intermediate result through the ratio 6:8, obtained by dividing the money by the points needed to win. Drawing on habitual techniques and wording (“(...) six euros divided by eight, which is the total, by how many points...”), she makes the use of proportionality knowledge recognizable.

When analyzing this turn, together with the rest of turns by the students of EA in their communication with the teacher, we cannot find explicit references to meanings attributable to randomness. This is in contrast to the fact that some of the reasons used by Anna in peer work (see the first exemplified turn) were related to tacit taken-as-shared knowledge concerning random games and randomness. In the interaction with the teacher, the traces of probability reasoning are not brought up and, therefore, the discussion of parts of this reasoning does not take place. Instead, Anna focuses her report on well-established techniques related to proportionality that emerged during peer work. She shares knowing-that with the teacher regarding proportionality techniques as they tend to be taught in secondary school mathematics, and she does not refer to the tacit knowledge and taken-as-shared knowing-how that have been enacted in her group to first attempt a model for the situation.

Our analyses show that this is an instance of a more general phenomenon, namely that epistemologically relevant aspects of the students’ group work are often not made visible in the interaction with the teacher. In this interaction, students tend to focus on well-established knowledge and techniques of the secondary school mathematics repertoire. Drawing on our micro-macro theoretical perspective, we find two levels of explanation for this phenomenon. On the macro-level, the change in the focus may be tied to the historical emphasis on the use and evaluation of knowing-that and techniques as taught by the teacher in secondary school mathematics. On the micro-level, we see how Anna adjusts her vocabulary in keeping with the use of “fair” by the teacher. Moreover, this student reports the group’s mathematical work by describing computational procedures (e.g. “we multiplied each point by zero point five”) without justifying their adequacy in the context of resolution of the problem.

FINAL REFLECTIONS ON MISSED OPPORTUNITIES

In Goizueta et al. (2014) and Goizueta (2015), the analyses of the students’ activity concerning the resolution of the proposed problem led to results about a diversity of knowing-that and knowing-how school mathematics practices. Since some of those practices were not totally developed during the lessons, the insufficient exploitation of certain learning opportunities –i.e. mathematics learning opportunities generated in the course of classroom interaction (Planas, 2014)– could be argued. In the present investigation, since some of the ideas and tacit knowledge in the students' reasoning in small groups are not opened up for discussion with the teacher, a number of missed learning opportunities can be similarly outlined. Any communication of the students' group work to the teacher is always necessarily incomplete and, therefore, not all the ideas happen to be shared, assessed and turned into mathematics learning opportunities. One question is whether there is a particular type of ideas concerning knowledge about the construction and justification of school mathematics knowledge, whose discussion tends to be underestimated in the mathematical culture of the mathematics classroom. What we have illustrated in this report may not be unique.

Regarding the context of interaction with the teacher, the students may have constructed some learning of the fact that sharing certain issues involved in their mathematical activity, and omitting others, can situate them better as learners of mathematics in that classroom and with that teacher. The students of EA may have intentionally omitted their approaches to probability arguments and ideas of randomness if they perceived that this part of their mathematical activity was irrelevant or not adequate in the public context of interaction with the teacher. If this is the case, important mathematics learning opportunities are missed due to representations of what is adequate in the mathematical culture of the mathematics classroom.

Acknowledgements

The authors' participation in this study has been funded by the Spanish Grant EDU2012-31464 and the Catalan Grant SGR2014-972. The second author has also been funded by the Catalan Institute of Research and Advanced Studies (ICREA).

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CHILDREN'S PATTERNS OF REASONING IN INTUITIVE MENTAL RATIO COMPARISON

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Understanding rational numbers requires rich notions of ratio and proportionality. Previous research has showed that children tend to reason exclusively based on natural numbers when asked to compare fractions, ignoring the involved ratios. The present work investigated whether this type of reasoning emerges in a context that emphasizes reasoning about ratios. Forty 2nd-grade children learned to use fraction-like symbols to represent ratios of candies received over a number of days, and were then evaluated on a ratio comparison task using those symbols. A clustering analysis revealed the presence of three groups of children deploying distinct patterns of thought including the reasoning purely based on natural numbers, revealing that this pattern of reasoning still arises in intuitive mental ratio comparison.

INTRODUCTION

Rational numbers are a crucial concept in elementary and middle school mathematics and learning them may be one of the most difficult achievements, requiring a conceptual shift regarding what numbers are. In line with this high difficulty, research has linked rational number understanding to both previous and future mathematics achievement (e.g. Aksu, 1997; Siegler et al., 2012). Still, many educators may not be prepared to teach fractions because of lack of understanding of the core concepts: Depaepe et al. (2015) showed that prospective teachers' knowledge of fractions reached an average of 79% (range 34%-98%) when tested with a questionnaire appropriate for upper elementary school according to the curriculum (see also Van Steenbrugge, Lesage, Valcke, & Desoete, 2014).

One of the first relevant questions to deal with for teaching rational numbers is how to approach them, with several possible interpretations or metaphors available to the educator. Rational numbers can be represented as parts of an object, as positions in the number line, or as ratio-based relations between two quantities, among others. Educators often emphasize one or two of these representations over the others—most commonly the parts-of-an-object one—, and as a result the view of fractions as parts of objects is very common, whereas ratio-based relations and a ratio-based approach to fractions tend to be introduced much later, or sometimes simply neglected. Although ratio and proportions are topics widely recognized as important and part of most mathematics curricula worldwide (see Obando, Vasco, & Arboleda, 2014, and references therein), their late introduction to students might miss the opportunity of taking advantage of children's early intuitive understanding about ratios (e.g. Singer, Kohn, & Resnick, 1997; Van Den Brink & Streefland, 1979).

Several researchers in Mathematics Education and Psychology have studied a phenomenon named Natural Number Bias (hereafter NNB), consisting on the overgeneralization of concepts and intuitions proper of natural numbers to rationals (e.g. Ni & Zhou, 2005; Gómez & Dartnell, 2015; Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Van Dooren, Lehtinen, & Verschaffel, 2015). This bias can be appreciated in students' responses to fraction comparison items where fractions share a common numerator (e.g. 3/5 *vs*. 3/7) or denominator (e.g. 5/9 *vs*. 7/9). Items with a common denominator are systematically found to be easier than items with a common numerator (e.g. Gómez et al., 2014; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Eeckhoudt, 2013), because the magnitude of the relevant components (5 < 7) points either in the opposite (3/5 > 3/7) or the same (5/9 < 7/9) direction as the fractions' magnitudes.

In line with previous studies, this paper will consider as congruent those fraction pairs in which the largest fraction is the one with the largest numerator and/or denominator, and incongruent those pairs where the largest fraction is the one with the smallest numerator and/or denominator (e.g. Gómez et al., 2014; Obersteiner et al., 2013; Vamvakoussi et al., 2012). As in these works, we also extend the notion of congruency to fraction pairs without common components: an example of a congruent item is 1/3 < 5/7, where the larger numerator and denominator both belong to the larger fraction, whereas an example of an incongruent item is 2/3 > 4/9, in which the larger numerator and denominator belong to the smaller fraction. The NNB has been found in the vast majority of studies about fraction comparison and fraction knowledge more generally, and traces of it seem to be even present in expert mathematicians' response times to compare fractions (Obersteiner et al., 2013). It is so far an open question the extent to which the emergence of the NNB depends on the pedagogical strategies or approach used to teach fractions (Ni & Zhou, 2005), although the concept of congruence itself has been recently problematized in terms of its cognitive relevance (Gómez & Dartnell, 2015).

The present study had two goals. First, to evaluate second grade children's ability to use their intuitive understanding of ratios to respond to a ratio comparison test akin to the fraction comparison tasks documented in the literature (e.g. Gómez et al., 2014). Second, by including in the ratio comparison test both congruent and incongruent pairs of ratios, we evaluated if a NNB similar to the one documented for fractions appears in this novel setting. To do this, a series of brief audiovisual recordings were developed and presented to second grade children in order to teach them how to represent ratios by using fraction-like symbols. These symbols (see Figure 1 for an example) were messages that represented the number of candies that children in a fictional world would receive over a number of days. This way, children were asked to compare ratios by judging which one of two of these messages was "more convenient". This context for the presentation of ratios falls into the format category of associated sets studied by Lamon (1993), consisting in creating ratios by pairing objects from two different sets—in our case, the set of candies and the set of incoming days—at a fixed rate. Lamon asked sixth-grade children (who had not yet received formal instruction about ratios) to answer problems about ratio and proportion in different formats. She found that problems presented in the associated sets format were most frequently solved by using some sort of qualitative proportional reasoning, in contrast to other formats that did not elicit appropriate proportional reasoning often.

This work thus aims at exploring the feasibility and utility of using children's intuitions about ratios in order to avoid the NNB and achieve successful comparison of ratios, as a possible way to improve the teaching of fractions and rational numbers in general.

MATERIALS AND METHODS

Participants

Forty Chilean 2nd-grade children (19 boys and 21 girls, about 7-8 years old) participated in this study. Signed informed consent for participation was obtained from a parent of each participant prior to the testing session.

Material

Audiovisual recordings. Five audiovisual recordings presented the story of a fictional land where elves bring messages and candies to children overnight. The reception of a message like the one in Figure 1 meant that the recipient would get two candies every five days. Two families of elves (yellow and green) were introduced, in order to give children the possibility of choosing which one of two messages was more convenient for them. The average length of each recording was 3:12.

Ratio comparison task. Children answered twelve ratio comparison items presented on the computer screen (see Figure 2). In addition to having both congruent and incongruent items, the task included also items in which the two ratios shared the same number of days or the same number of candies.

Procedure

Children were tested in the computer classroom of their school, in groups of 20 to 30 children. Each child worked individually with a computer in a single testing session. After an introductory explanation of the content of the session, children watched the five audiovisual recordings. Each video was followed by two questions used to probe children's understanding of specific contents.

At the end of the session, children answered the ratio comparison task. For each item, they were presented with a pair of ratios and asked to judge which was "more convenient" by pressing the keys Q or P. Children had no time limit for answering.

Data analysis

We analysed only accuracy data for the ratio comparison test. Children were grouped in clusters using the *k*-means clustering algorithm (e.g. see Steinley, 2006) by considering their responses to each of the four item types separately. The value of *k* was chosen as the largest possible not generating clusters of size 5 or smaller.

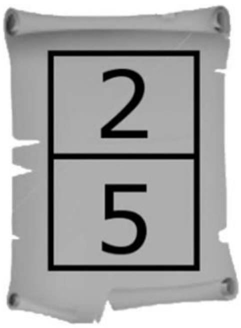

Figure 1: Example of a message used to represent ratios, meaning that its recipient would receive two candies every five days.

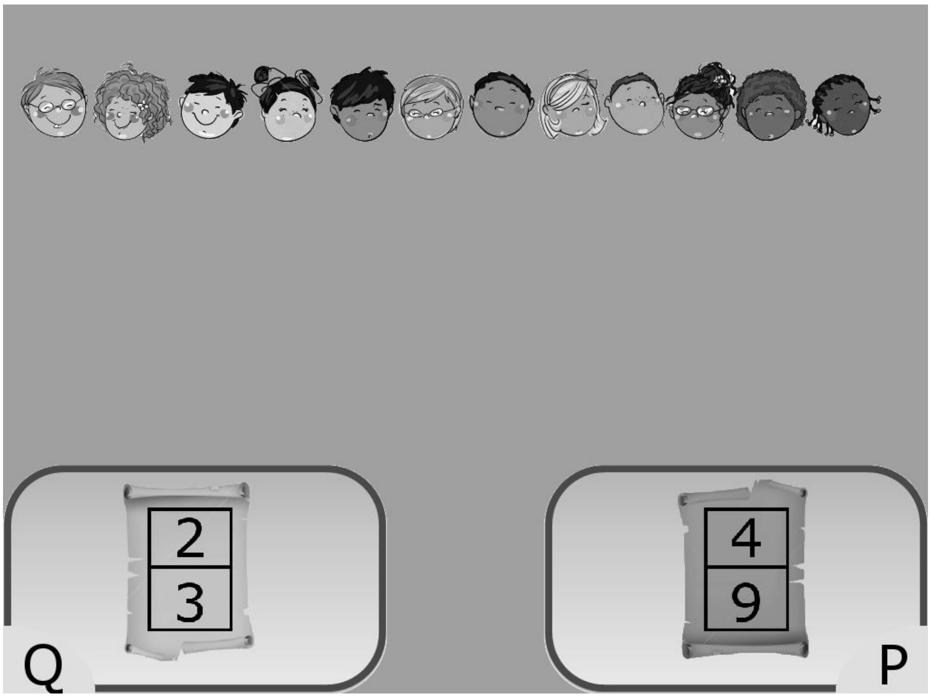

Figure 2: Screen capture of an item of the ratio comparison test. The children depicted at the top of the screen switched from grayscale to color and started smiling after the participant responded to each item, regardless of the correctness of the given answer.

RESULTS

The full sample of children had a mean accuracy of 65% or approximately 8 correct answers out of 12. This value is modestly but significantly above the chance level of 50% ($t(39) = 5.4$, $p < .0001$). This suggests that, at least in average, children were able to adequately interpret the symbols used for ratios. Table 1 displays separate scores for each of the four item types. These scores were analyzed by means of a logistic regression with congruency and the presence/absence of a common component as fixed factors and children as a random factor. This regression revealed a significant main effect of congruency ($OR = -1.0$, $p = .0005$) where congruent items were answered more correctly than incongruent items (average scores of 76% and 54%, respectively);

a trend towards significance for the main effect of the presence/absence of common components (*OR* = 0.58, *p* =.06) where items sharing a common component were answered more correctly than items with all numbers different (average scores of 70% and 60%, respectively); and no significant interaction between these two factors (*OR* = -0.15, *p* = .72).

We then analyzed individual differences in children's understanding by means of a *k*-means clustering analysis, considering accuracy scores for the four item types for each child. This analysis revealed the existence of three groups of children explaining 67.3% of the total variance. Table 2 shows the scores of each group and item type.

Cluster A was the largest, with 17 out of the 40 children. This cluster also had the highest overall score (77%). Children in this cluster showed high scores for all item types except for incongruent pairs with no common components.

The second most numerous group was Cluster B, with 14 out of 40 children. Cluster B comprised children who answered mostly guided by the congruency or incongruency of each item according to the NNB account. That is to say, children in this group had very high scores in answering congruent items and very low scores in answering incongruent ones. Hence the overall score of this group was markedly lower than the other groups (48%).

Group	With a common component		Without common components		Total score
	Congruent	Incongruent	Congruent	Incongruent	
Full sample (N = 40)	81 %	59 %	71 %	49 %	65 %
Cluster A (n = 17)	82 %	80 %	84 %	63 %	77 %
Cluster B (n = 14)	83 %	7 %	95 %	5 %	48 %
Cluster C (n = 9)	74 %	100 %	7 %	93 %	69 %

Table 1: Average accuracy per item type for the full sample and clusters. Congruent items with a common component are those in which the two messages share the same number of days, whereas incongruent ones are those in which the two messages share the same number of candies.

The third and last cluster, Cluster C, was composed by 9 out of the total 40 children. They exhibited a good overall score (69%), but this score conceals an unexpected pattern of responses: Incongruent ratio pairs were answered almost completely correct, congruent items without common components mostly incorrect, and surprisingly, scores in the easiest item type (congruent items with a common component) were not as high as expected.

DISCUSSION

The present research aimed at understanding whether children's intuitions about ratios can help them to respond successfully to a ratio comparison test, and to explore if this approach elicits a NNB such as the one many studies have documented for fractions at different ages and levels of expertise (Gómez et al., 2014; Obersteiner et al., 2013; Vamvakoussi et al., 2012; Van Eeckhoudt, 2013). The data shed light on both questions. Children's intuitive reasoning seemed to be, in average, a good scaffold for answering the ratio comparison task. Nonetheless, a clustering analysis revealed that children showed distinct patterns of intuitive reasoning, not all of them being compatible with an adequate concept of ratio.

A majority group (cluster A, 43% of the sample) compared ratios successfully across all item types, indicating that intuitive reasoning via ratios may be successful as a pedagogical tool for introducing ratio and proportion. Still, 35% of children (cluster B) reasoned mostly based on the natural numbers composing the ratios (a strong form of NNB, cf. Gómez et al., 2014), disregarding the relations between numbers of candies and days presented in each message and focusing only on comparing the corresponding numbers across the two presented messages. Although we did not ask children to justify their decisions, they would have probably given explanations where ratios simply consist in two independent numbers. Stafylidou and Vosniadou (2004) presented and considered this explanatory framework as the most basic one, showing that it is still used by 30% of children in 5th grade. Finally, the behavior of children in the third group (22% of the sample) also departed from the predictions of the NNB. If anything, they showed a reversed bias: scores for incongruent items were higher than those for congruent items (in opposition to, e.g., Gómez et al., 2014; Van Eeekhoudt, 2013; but see Gómez & Dartnell, 2015). This pattern of answers suggests that they might have focused exclusively on the number of days presented in each message, choosing whenever possible the message with the smallest number of days associated. A possible account for this group's reasoning might go beyond ratio comparison *per se*, as they seemingly deemed more convenient to choose that candies are delivered sooner even if that means receiving fewer candies. Such possibility suggests that these children interpreted the ratio comparison task as an economical decision rather than anything ratio-based (indeed, children facing such decisions seem to prefer shorter delays than higher rewards, e.g. Green, Fry, & Myerson, 1994). It is not uncommon that children use different interpretations or intuitions than those expected by educators and researchers (e.g., Van Den Brink & Streefland, 1979), which leads to one limitation of the present study: the absence of individual interviews or similar methods of inquiry allowing confirmation of how children were actually reasoning. Still, the cluster-based analysis provides an important first step in that direction by grouping children according to their patterns of answers. Further research is also needed to discover whether children's clusters of membership predict how they will reason about fractions later, and to explore continuities and discontinuities in the transition from intuitive ratio concepts to formal ones.

These results also have implications for the teaching of ratios and fractions. First, biased children (those in cluster B) might benefit from highlighting the relevance of integrating both natural numbers within each ratio into a holistic element and explicit discouraging of simple comparison of the natural numbers across ratios. Hence, activities such as mapping ratios and fractions onto a number line might prove useful for them. Second, the data showed that the same teaching material can be interpreted in a diversity of ways by children, even when overall performance seems satisfactory. The average score of children in cluster C was not substantially lower than that of children in cluster A, but looking at their patterns of answers revealed that they resorted to radically different reasoning paths, demonstrating that a good average score does not guarantee a correct understanding of the intended concepts. This highlights the relevance of improving the assessment of mathematics understanding, and in particular the need of presenting and explaining ratios and fractions using a variety of ways and/or metaphors, so as to minimize the chance of children drawing wrong interpretations or generalizations.

Acknowledgements

The authors are grateful to the students, parents, teachers, and school personnel who participated in this work. This research was funded by CONICYT PIA/Basal grant FB0003.

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THE ROLE OF THE KNOWLEDGEABLE OTHER IN POST-LESSON DISCUSSIONS IN LESSON STUDY

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Worldwide interest in Japanese Lesson Study as a vehicle to improve mathematics teaching practice through professional learning has left largely unanswered questions about the extent to which it can be replicated elsewhere. This paper reports some of the findings from a small-scale research project, "Implementing structured problem-solving mathematics lessons through lesson study", carried out in three Australian schools during 2012, and continued in a modified form during 2013 and 2014. In particular, it discusses the potential contribution to teacher professional learning resulting from post-lesson discussion commentaries by "knowledgeable others" with considerable experience of and expertise in lesson study within and outside of Japan.

INTRODUCTION

Lesson study first came to worldwide attention through Yoshida's (1999) doctoral dissertation and Stigler and Hiebert's (1999) accounts of Japanese "structured problem-solving" lessons based on the *Third International Mathematics and Science Study* (TIMSS) video study. These structured problem-solving lessons represent a major Japanese instructional approach designed to develop mathematical concepts and skills through problem solving (Takahashi, 2008).

Since 1999, there has been phenomenal growth of lesson study as a vehicle for teacher professional learning ouside of Japan. However, worldwide interest in, and attempts at implementation of, Japanese Lesson Study have left largely unanswered questions about the extent to which it can be replicated elsewhere, with research suggesting that teachers outside Japan often focus only on superficial aspects of lesson study (e.g. Robinson, 2007; Perry & Lewis, 2008).

Despite its long-standing tradition in Japan, research into lesson study only commenced in Japan as a result of the interest shown by Western nations, with Fujii (2014) noting that in Japan "lesson study is like air, felt everywhere ... [but] so natural that it can be difficult to identify its critical and important features" (p. 66). Moreover, according to Takahashi and McDougal (2016), early research articles based on case studies describing Japanese Lesson Study do not explain which parts of the process are essential and which could be modified, with "important aspects of lesson study as practiced in Japan … getting 'lost in translation'" (p. 2). Elsewhere, while many studies of professional learning programs have provided evidence of improved teacher learning, few have differentiated critical elements that have contributed to this learning.

Lesson study is neither an end in itself, nor about perfecting a particular lesson, but instead is a process through which teachers can gain new knowledge for teaching.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 315–322. Szeged, Hungary: PME.

Based on their experience of lesson study in Japan and USA, Takahashi and McDougall (2016) define collaborative lesson research (CLR) as having six components: a clear research purpose; *Kyouzai kenkyuu* (the study of materials for teaching); a written research proposal; a live research lesson and discussion; the involvement of "knowledgeable others"; and the sharing of results.

Misconceptions regarding the nature of lesson study and modifications to its implementation outside Japan have been found to include: repeated revisions and reteaching of the same lesson; entire lesson study cycles being carried out over the course of one day; a lack of observers beyond the planning team, with possibly just one other teacher being present at the research lesson and the post-lesson discussion; and an absence of knowledgeable others (Fujii, 2014; Takahashi & McDougal, 2016). In particular, Takahashi and McDougal draw attention to the importance of the role of knowledgeable others to support professional learning, echoing Guskey and Yoon's (2009) findings from their synthesis of research into professional development that the "efforts that brought improvements in student learning focused principally on ideas gained through the involvement of outside experts" (p. 496).

In Japan, lesson study almost always includes a knowledgeable other with in-depth experience of lesson study, and knowledge of curriculum and teaching, who offers final comments at the post-lesson discussion, while ideally there is a second knowledgeable other involved in the planning phase.

This paper reports on some of the findings from an Australian research project, *Implementing structured problem-solving mathematics lessons through lesson study*. Using data from a series of post-lesson discussions from research lessons planned and taught by Grades 3 and 4 teachers participating in this project, it looks at the potential contribution to teacher professional learning resulting from the final commentaries of knowledgeable others with considerable experience in acting in this capacity both within and outside of Japan.

THE PROJECT

The *Implementing structured problem-solving mathematics lessons through lesson study* project was carried out in three Australian elementary schools during 2012 to explore ways in which key elements of Japanese Lesson Study could be embedded into Australian mathematics teaching and professional learning.

Six Grade 3 and 4 teachers and four numeracy coaches[1] took part in an initial whole-day professional learning session on Japanese Lesson Study, after which they were divided into two cross-school teams, with each team conducting two research cycles. Participants planned each research lesson during four two-hour sessions. Two researchers joined each planning team and acted as knowledgeable others to facilitate and support, but not lead the planning.

[1] Numeracy coaches are experienced teachers who provide curriculum leadership in mathematics teaching by working with teachers to improve teaching and student learning.

One teacher from each team taught the research lesson. Members of both planning teams, key staff at each school, together with all interested teachers who could be released from their classes, observed the research lessons and took part in the post-lesson discussions. In all between 20 and 30 people – including members of the leadership teams from other schools, staff from the regional office, mathematics educators, and a knowledgeable other – observed each research lesson and took part in the post-lesson discussions. Every attempt was made to implement Japanese Lesson Study as "authentically" as possible, including adopting the structured problem-solving pattern for the research lessons. The knowledgeable others giving the final comments were Dr Max Stephens, an Australian expert on Japanese Lesson Study, and our Japanese collaborator, Professor Toshiakira Fujii, who came to Australia to act as the knowledgeable other for two of the 2012 research lessons. The project continued in a modified form in 2013 and 2014.

During 2012, all planning sessions, research lessons, and subsequent post-lesson discussions were video recorded. Three audio-recorded, semi-structured interviews were also carried out with each of the participants. Field-notes, lesson plans, and other artefacts, such as student work, were also collected. Similar data were collected during the two research cycles in 2013, although in this case there was just one interview. The only data available from the 2014 mathematics lesson study cycles were video recordings of the first and third mathematics research lesson and the subsequent post-lesson discussions, together with lesson plans and student work.

All interviews and post-lesson discussions were transcribed. This paper is based on a thematic analysis of these transcripts, and field-notes from the planning sessions.

THE ROLE OF THE KNOWLEDGEABLE OTHER

Takahashi (2014) used a case study of three respected knowledgeable others in Japan to clarify the role of the knowledgeable other in lesson study, and argues that in Japan such expertise comes from years of experience participating in lesson study, which is often difficult to replicate outside of Japan. In this section we provide **exemplars of comments from the knowledgeable others in our project and discuss how these comments had the potential to support various aspects of professional learning.**

Linking tasks to the aims for the lesson. Watanabe, Takahashi and Yoshida (2008), remind us that the purpose of lesson study is not just to improve a single lesson, but to improve mathematical instruction in general. This involves careful attention to *kyozai kenkyuu*, something that is not always attended to in non-Japanese lesson study. They further remind us that the same subject matter can be explored using different tasks, while different subject matter can be investigated with the same task – the important thing being to link the task to the aims of the lesson.

During his concluding remarks on Research Lesson 3 (RL3), Toshiakira Fujii (TF), our knowledgeable other for Research Lessons 3 and 4, highlighted the need for the aim of the lesson to be the driving force behind the choice of the task, as well as the need for a clear mathematical aim.

> So first of all the task. The task must be open-ended and rich enough in mathematics, also educational value there, but not too open, not too open ... otherwise children's flexibilities are great, they can go anywhere. Today's evidence shows us they can go anywhere. So [it] should be open-ended but narrowed down a bit, according to the aim of this, today's lesson. … So today the lesson's aim listed about six [aims]. I thought it too many ... Third one "choose and use learned facts procedures, strategies to find solutions". How about this one? (TF, RL3)

In Japan, even the numbers used in a particular task are contested and require deep consideration (see, for example, Doig, Groves, & Fujii, 2011). This was evident in another of TF's comments during the RL3 post-lesson discussion.

> Okay, I mean, yesterday we talk about the number, about task [with the research team]. The first I saw the task I immediately thought, oh 23. That means not 24. ... So why you choose 23 instead of 24? You could choose 24 x 2; I mean no carrying over, two digits times one digit. ... why you avoid to use 24 then to accomplish this task, this aim. ... So why they … choose 23 then? … [You] have to consider the role of the task, it should be open ended but should be related to the end. So if the team could answer, why not 24? ... But this task is 23, the prime number, why prime number? (TF, RL3)

Teaching through problem solving. The *Australian Curriculum: Mathematics* (ACARA, n.d.) identifies *Understanding*, *Fluency*, *Problem Solving*, and *Reasoning* as the four strands that comprise mathematical proficiency. However, the focus in *Problem Solving* is on the "process" aspects, with no suggestion that mathematical content can or should be taught *through* problem solving. By way of contrast, in Japan the problem-solving lesson structure for mathematics, which has evolved over four decades, originated in a desire to introduce open-ended problems in order not only to enhance students' higher-order thinking, but also to enable students to use their previous knowledge and skills to learn something new through the process of solving a problem (Becker, Silver, Kantowski, & Wilson, 1990).

During his comments on RL3, TF highlighted the need to raise students' levels of understanding in the discussion (*neriage*) phase of structured problem-solving lessons.

> Yes, well let's talk about the problem-solving orientated lesson. There's two critical parts. One is a task, and ... second is the discussion period. ... [It] means starting from children's level and [raising] their thought. That's the most difficult part in a problem-solving lesson. (TF, RL3)

Takahashi (2008) describes this *neriage* phase as the heart of the lesson and the starting point for student learning through the teacher highlighting "important mathematical ideas and concepts for students to reach the goals of the lesson" (p. 5). Solving the problem and sharing solutions is the beginning of the learning process rather than the outcome at the end of the lesson.

Educational goals versus process goals. An area where teachers in our project struggled was in formulating learning goals – especially ones that related to mathematical learning rather than observable outcomes such as students finding and communicating multiple solutions to the problem or "demonstrating confidence".

> I found it surprising how hard it was for us to articulate what the goals of the lesson were going to be. I thought it would be pretty easy ... [but] I was surprised how long we spent trying to articulate [these]. (George, Interview 3)

In Japan, structured problem-solving lessons typically include both content goals and goals relating to problem-solving skills and strategies, with Takahashi (2008) describing these lessons as vehicles for introducing new topics or big ideas. TF commented on this after RL4.

> So Japanese lesson study's aim is not to solve this task at all – problem-solving oriented lesson means through solving this task we want to teach how to think. ... that's why let them think first then discuss how you thought ... so the discussion is not only how to solve this one [task]. It should focus on the thinking itself. (TF, RL4)

He further commented on not only how difficult this is, but also how important it is if we are to take seriously our roles in educating the whole child.

> That is difficult. You should know mathematics, you should know children's way of thinking. Therefore teacher is professional job. You see you should be honoured to be a teacher ... Skill is important ... [but we need] to educate children as a human being how to think. (TF, RL4)

Engaging students in whole-class discussion. While sharing of student solutions at the end of a lesson is common in Australian classrooms, it is often just a very brief "show and tell" (c.f. Takahashi, 2008). There is very little history of whole-class discussion, which itself is often equated with expository teaching. Teachers in the project went to considerable lengths to overcome practical difficulties associated with conducting extended whole-class discussions – for example, the classroom layout that necessitated students sitting on the floor for these discussion – and students responded well to being asked to explain their solutions during the orchestrated discussion. However, moving beyond to-and-fro interactions between the teacher and individual students proved difficult.

Max Stephens (MS), our knowledgeable other for Research Lessons 1, 2, 5, 6 and 7, offered advice on this aspect.

> A lot of the discussion was between Megan[2] and a student. In a problem-solving lesson there is a place for asking the students "Do you have a question to ask of Holly?" I'd throw a bit of responsibility back on the students ... I think it is very helpful to [ask the other students] "Do you have a question about what Holly has said that wasn't clear?" … So they need to own [it]. Otherwise I think it was a little too passive. But it was a very usable phase. (MS, RL5)

> And it comes back again in the sharing of solutions – which I think was very well done. But I would encourage you in the sharing of solutions to occasionally throw it out to the whole class. Very clear dialogue between you and the student presenting, but occasionally

[2] All teacher and student names are pseudonyms.

> you [could] say to people "Have you got a question?" or "What are you noticing?" It wouldn't make the lesson much longer but it means that those who haven't been called up are drawn into the discussion. (MS, RL7)

When making similar comments after RL6, MS emphasised the fact that teaching students how to participate in genuine mathematical discussions was a whole-school responsibility and that you could not expect them to suddenly to know how to do this in Grade 3. These comments were not merely addressed to the teacher of the research lesson, or even the entire planning team. Instead they were addressed to all of the observers present at the research lesson and were later reported anecdotally by the Assistant Principal as having led to a number of "corridor conversations" amongst teachers.

Concluding his remarks on RL3, TF encouraged teachers to persist in the difficult task of "moving the discussion up one level", emphasising the value of observing research lessons and participating in the subsequent post-lesson discussions.

> Focusing on discussion period, ... we could move level up you see. ... In Japan we struggle to do that, it's very difficult, so we communicate with each other and keep studying each other, we learn from each other. Thank you. (TF, RL3)

Participants' views. Participating teachers highly valued the post-lesson discussions and regarded the input from the knowledgeable others as important learning opportunities.

> What I've noticed is, sort of the culture that sits behind it. I can imagine the Japanese post-lesson discussions being a lot more, as I said honest, really perhaps a little bit more animated, a bit more challenging. I think we're very polite here. And I know Japanese are polite, but I have a feeling they're a bit more blunt in this context. So … that whole thing of changing the culture at the school, to support this model, would result in the best possible discussions, that culture of giving and receiving feedback. (Narah, Interview 3)

> The post-lesson discussions have been so valuable. Both people were really valuable people to listen to and had real insights and very deep knowledge of the content and classrooms. (Paula, Interview 3)

> Having someone like [TF] here today ... when he was speaking everyone was listening ... I like his approach. I know with sport ... that's the feedback I like. ... it can be harsh at times and you probably don't want to hear what some of them are saying. But these people are here for a reason – like you guys haven't invited any Joe Blow off the street to come and give advice. (Trevor, Interview 3)

DISCUSSION AND CONCLUSION

As discussed earlier, many adaptations of Japanese Lesson Study omit critical features of lesson study. In particular, the emphasis on and perceived success of collaborative planning often overshadows the value of the "open" research lesson and post-lesson discussion, which Lewis and Tsuchida (1998) regard as an essential component of lesson study. They quote one teacher as likening a lesson to "a swiftly flowing river" with observers' comments revealing "your real profile as a teacher … for the first time" (p. 15).

In this paper, we have highlighted the ways that experienced knowledgeable others have the potential to contribute to teachers' professional growth. We argue that Japanese Lesson Study is a powerful model for professional growth for all participants, with the learning that happens through this process not being restricted to the lesson planning team or the teacher teaching the lesson. Instead, as Archer, et al. (2013) point out, "The strength [of lesson study] is in the rigour of the post lesson analysis and that this feeds directly into the practice of all teachers present. The lesson studies are also significant learning opportunities for all others involved, particularly when there are visitors from Japan joining the lesson study" (p. 5).

Lesson study in Japan is widely viewed as a shared professional culture that provides a pathway for continuing improvement of teachers' pedagogical and content knowledge. However, as Stigler and Hiebert (1999) point out, efforts at improving teaching often ignore the fact that teaching is a cultural activity, which implies gradual change and the need to take into account the cultural assumptions underpinning teaching and learning. This raises many issues regarding the sustainability of Japanese Lesson Study as a model for teacher professional learning in Australia. While project teachers frequently referred to their experience of lesson study as being their most valued professional learning, further research is required to establish the development of sustainable lesson study groups within the Australian school culture – ones that do not rely on the degree of support provided to the project schools. This will necessitate the establishment of a community of teachers, mathematics educators, and researchers who can continue the process, including providing initial exposure to lesson study and acting as knowledgeable others.

Acknowledgement

The *Implementing structured problem-solving mathematics lessons through lesson study* project was funded by Deakin University. The project team were Susie Groves, Brian Doig, Colleen Vale and Wanty Widjaja, in collaboration with Professor Toshiakira Fujii, Tokyo Gakugei University.

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MODELING THE RELATIONSHIPS AMONG SOME ATTITUDINAL VARIABLES TOWARD MATHEMATICS

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In the field of affect many researchers have underlined to clarify constructs such as beliefs, emotions and attitudes, and to better investigate the relationships among them. In our previous study we tested the tripartite model of attitude, according to which attitude has a cognitive, an affective and behavioral component via structural equation modeling and found significant results. In this paper we hypothesize two additional structural models that investigate further relationships using the same data for the previous study. We will present the results and comment upon them.

INTRODUCTION

It is commonly accepted that affective factors play crucial roles in mathematics learning. There are many studies that have been able to establish a relationship between attitude and achievement in mathematics. However, many researchers have highlighted the need of some theory for research on affect, in order to better clarify connections among the various components, and their interaction with cognitive factors in mathematics education (McLeod, 1992).

In mathematics education, there is a variety of definitions of the term attitude. Using a multidimensional definition, attitude toward mathematics comprises three components: a conception about mathematics, an emotional respond to mathematics, and a behavioral tendency with regard to mathematics (Hart, 1989). This definition has gradually been recognized at a theoretical level (Di Martino & Zan, 2011), a *tripartite model*, according to which attitude has a cognitive, an affective and a behavioral component. In our previous study we accepted this tripartite model of attitude and hypothesized a second-order factor model where attitude is a single second-order factor; cognitive, affective and behavioral components are second-order factors. In this study we continue at that point and hypothesized further relationships.

In assessment of attitude, the Fennema-Sherman Mathematics Attitudes Scales (FSMAS) (Fennema & Sherman, 1976) remain the most extensively used in research studies (Hyde et al., 1990). The FSMAS comprise nine scales: Attitudes towards Success in Mathematics, Mathematics as a Male Domain, Confidence in Learning Mathematics, Mathematics Anxiety, Effectance Motivation in Mathematics and Usefulness of Mathematics. They also include Mother, Father and Teacher scales. The subscales can be used as a set, or individually. Recent studies have generally provided support for the reliability and validity of the FSMAS (Melancon et al., 1994). In our previous work we used some of the FSMAS and adapted two more scales all represent three components of attitude toward mathematics. In the present study we hypothesized

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 323–328. Szeged, Hungary: PME.

two additional structural models including such variables. The research questions of the study were:

- What is the model explaining relationships among students' perceptions of their mathematics teacher's teaching profession, their mathematics teacher's, father's and mother's attitudes toward and expectations from them as learners of mathematics, their confidence in learning mathematics, beliefs about the usefulness and importance of mathematics, liking for mathematics, mathematics anxiety, behaviors toward mathematics and the time they spent on mathematics at home?
- What is the model explaining relationships between students' perceptions of their mathematics teacher's teaching profession, their mathematics teacher's, father's and mother's attitudes toward and expectations from them as learners of mathematics and cognitive, affective and behavioral components of attitude toward mathematics?

METHOD

Sample

The sample of the study consisted of 1960 7th grade students enrolled in 19 different public elementary schools in one of the districts of one of the big cities of Turkey. Convenience-sampling was used to select the subjects. In the total sample, 1001 (51.1%) students were female and 959 (48.9%) students were male.

Instrument

In order to test the hypothesized models, Attitude Toward Mathematics Questionnaire (ATMQ) was used. It involves ten scales: confidence in learning mathematics (12 items), usefulness and importance of mathematics (16 items), liking for mathematics (5 items), mathematics anxiety (12 items), learner behaviors toward mathematics (4 items), time spent on mathematics at home (4 items), father scale (11 items), mother scale (11 items), teacher scale-I (12 items) and teacher scale-II (7 items) (94 items in total). It is scaled on a five-point Likert type: strongly agree, agree, undecided, disagree, and strongly disagree. The subscales confidence in learning mathematics, usefulness of mathematics, liking for mathematics, mathematics anxiety, father, mother and teacher-I were adapted from the corresponding subscales of the FSMAS (Fennema & Sherman, 1976) by Tag (2000); importance of mathematics and teacher-II were adapted from TIMSS (1999) by Tag (2000); learner behaviors toward mathematics was adapted from the questions in 'student interview guide' developed by Beth and Neustadt (2005); and time spent on mathematics at home was adapted from the statements of the instrument developed by Mohamad-Ali (1995).

Preliminary data analyses for the instrument were done to detect the outliers, check the data recording (data cleaning) and normal distribution of the variables. The alpha reliability coefficients for ten subscales were found 0.879, 0.878, 0.769, 0.827, 0.599, 0.659, 0.843, 0.840, 0.690 and 0.724, respectively. To test the construct validity of each subscale and determine whether or not they have sub-dimensions, principle component analysis was done. A confirmatory factor analysis with ten factors was carried out to assess the fit using LISREL. All the fit indices indicated that the model proposed fitted to the data set.

RESULTS

For testing the proposed models, covariance matrixes were generated using PRELIS. Significance of the path coefficients was tested through *t*-tests. Maximum likelihood estimation was used for estimating parameters of the models.

Results of the First Research Question

In order to test the proposed relationships among variables, a path analytic model was hypothesized. Initially, to revise the hypothesized model data fit, the selected LISREL-SIMPLIS model fit indices and the significance of the paths was considered with respect to the *t*-test results. In addition, the modification indices were checked and covariance terms were added if needed. The path between students' perceptions of their mother's attitudes toward and expectations from them as learners of mathematics and mathematics anxiety was found to have non-significant *t*-value; therefore it was removed from the hypothesized model. Moreover, as a result of inspecting the modification indices, covariance terms were added between eight pairs of observed variables. As a result, all the goodness-of-fit indices of the model were investigated through their criteria and because of RMSEA, it was concluded that the model indicated a poor fit to the data ($\chi2$ = 443.55, p = .00, df = 14; GFI = .96; AGFI = .83; SRMR = .063; RMSEA = .13). However, the relationships among the variables of the proposed model were examined using correlation analysis. Table 1 shows the inter-correlations of the variables used.

	CO	UI	LIKE	ANX	MBEH	TIME	TETP	TEST	FAST
UI	.54								
LIKE	.62	.62							
ANX	.69	.48	.59						
MBEH	.53	.51	.50	.45					
TIME	.45	.45	.46	.41	.44				
TETP	.36	.48	.40	.32	.37	.27			
TEST	.50	.48	.44	.52	.40	.39	.41		
FAST	.40	.52	.37	.39	.48	.39	.31	.44	
MOST	.42	.54	.38	.36	.51	.42	.32	.45	.71

Note. CO=Confidence in learning mathematics, UI=Usefulness and importance of mathematics, LIKE=Liking for mathematics, ANX=Mathematics anxiety, MBEH=Learner behaviors toward mathematics, TIME=Time spent on mathematics at home, TETP=Students' perceptions of their mathematics teacher's teaching profession, TEST=Students' perceptions of their mathematics teacher's attitudes toward and expectations from them as learners of mathematics, FAST=Students' perceptions of their father's attitudes toward and expectations from them as learners of mathematics, MOST=Students' perceptions of their mother's attitudes toward and expectations from them as learners of mathematics.

p<.01

Table 1: Inter-correlations of the variables

From Table 1, all correlations are positive and significant at the .01 level of significance.

Results of the Second Research Question

In order to investigate the second research question, a path analytic model with latent variables was hypothesized. In the model, *Cognitive Component of Attitude toward Mathematics*, the *Affective Component of Attitude toward Mathematics* and the *Behavioral Component of Attitude toward Mathematics* latent variables were defined by their respective observed variables. Same procedures were done for revising the hypothesized model data fit. The paths between students' perceptions of their father's attitudes toward and expectations from them as learners of mathematics and affective component of attitude; students' perceptions of their father's attitudes toward and expectations from them as learners of mathematics and behavioral component of attitude; and students' perceptions of their mother's attitudes toward and expectations from them as learners of mathematics and cognitive component of attitude indicated non-significant *t*-values. Therefore they were removed from the hypothesized model. Moreover, covariance terms were added between six pairs of observed variables. As a result, the final model fit indices indicated that our hypothesized path analytic model with latent variables has an acceptable fit ($\chi 2$ = 110.67, p = .00, df = 15; GFI = .99; AGFI = .96; SRMR = .017; RMSEA = .057). The strength and direction of the relationships among exogenous and endogenous variables were identified by γ (lowercase gamma) values and the structural equations of the model fitted for 1960 seventh grade Turkish students were obtained.

DISCUSSION

In the present study, when the models obtained were compared with the hypothesized models at the beginning of the study, it was seen that some of the proposed relationships were validated and some of them surprisingly did not. For example, in the second model obtained, no relationships were found between cognitive component of attitude toward mathematics and students' perceptions of their mother's attitudes toward and expectations from them as learners of mathematics; affective component of attitude toward mathematics and students' perceptions of their father's attitudes toward and expectations from them as learners of mathematics; and behavioral component of attitude toward mathematics and students' perceptions of their father's attitudes toward and expectations from them as learners of mathematics. Although no specific finding was obtained in the previous studies investigating the relationships between the students' perceptions of their teacher's and parents' attitudes toward them and three components of attitude; there are evidences in the literature that students' perceptions of their teacher's (Aiken, 1970; Kulm, 1980; Leder, 1992; Haladyna et al., 1983) and parents' (Eccles et al., 1983; Fennema & Sherman, 1976) attitudes toward and expectations from them as learners of mathematics had effect on their attitudes toward mathematics.

The findings of the present study indicated that further research should be conducted to examine the structure of attitude toward mathematics in terms of cognitive, affective and behavioral components. The influence of teacher's and parents' attitudes and expectations on three components of attitude should also be investigated. The models presented for attitude toward mathematics in this study had implications for further research studies.

Based on both the findings of this study and the related studies in the literature some implications for research methodology can be drawn. The first improvement needed in future research is the need to go beyond simplistic positive-negative distinction of affect. In this study, differentiating attitude toward mathematics as cognitive, affective and behavioral is very remarkable. Many of the mathematics attitude scales that have been constructed and used in research studies are generally intended to assess factors such as liking/disliking, usefulness, confidence. The choice of using items only about beliefs or emotions does not take into account the behavioral component. What seems to be implicit in this choice is the assumption that an individual's behavior toward an object has not got any meaning about his or her attitude toward that object. Therefore, in order to assess an attitude, we have to take into account all three components of it namely, cognitive, affective, and behavioral components.

Regarding affective traits, there is a need for new longitudinal studies with measurement instruments that would take into account the synergistic relationships between cognition, emotion, and behavior. Since simple answers cannot satisfy the complexity of classrooms, more attention should be paid to three main elements in order to study affect in mathematics education: cognition, emotion, and behavior. It is highly recommended that the researches on affect in mathematics classrooms should involve three approaches (observations, interviews, and questionnaire) which focus on emotional reactions of students in mathematics classes and achieve methodological triangulation.

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HOW CHINESE STUDENTS' PRE-SCHOOL NUMERACY SKILL MEDIATE THE EFFECT OF PARENTS' EDUCATIONAL LEVEL ON THEIR LATER MATHEMATICS ACHIEVEMENTS

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This paper presents a 15-year longitudinal study of how the Chinese students' mathematics achievements could be predicted by their parents' educational level, and their pre-school numeracy skill. Sixty-three Chinese children from Beijing, and their parents were participated in this study. It was found that both the parents' educational level measured at children's 1-year-old age and children's 5-year-old age pre-school numeracy skill can significantly predict students' mathematics achievements in high school when they were 15 years old. Furthermore, the effect of parents' educational level was mediated by the students' pre-school numeracy skill. Findings suggested that students' pre-school numerical skill is crucial for later mathematics achievements, and may be a protector for children in low-SES families.

INTRODUCTION

Students' mathematics achievements never fail to be the center of attention for parents, researchers, educators, and policy makers. As suggested in the Survey of Adult Skills (OECD., 2013), mathematics achievements have a major impact on individuals' life chances, influencing their ability to gain higher social prestige, and to have better physical and psychological health later in their lives.

The socioeconomic status (SES) is one of the most important family environmental factors to predict students' academic achievements. Students generally perform better in school if their SES-background is higher (Sirin, 2005; Wang, Li, & Li, 2014; White, 1982). In China, students' SES exerts significant influence on their mathematics achievements, and parents' educational level and income stand out among other factors (Wang et al., 2014).

Chinese parents are usually involved into pre-school educational activities. However, since parents in China seldom participate in children's mathematics learning activities, especially after primary school, the long-term effects of SES remain unclear. In a study with a sample of 532 children, Anders et al. (2012) argued that family SES significantly influenced early numeracy skills. In addition, some longitudinal studies have already demonstrated that early numerical skills such as object counting, number knowledge, nonverbal calculation and number combinations predict later mathematical performance at school (Aunio & Niemivirta, 2010; Hannula-Sormunen, 2015; Krajewski & Schneider, 2009a, 2009b).

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 329–336. Szeged, Hungary: PME.

Regarding the possible relationships among SES, early numeracy skills, and high school mathematics achievements, our hypothesis in the present study is that numeracy skills mediate the effect of SES (parents' educational level or income) on Chinese children's mathematics achievements. Although previous studies have tried to detect the effect of SES and early numeracy skills on students' mathematics achievements, few researches combined the three aspects in one study and focus on Chinese sample. The time spans for these studies were also too short to cover both the pre-school characteristics and high school achievements. To deeply detect the causal influence of SES and numeracy skill, a 15-year longitudinal design was used here to reveal the SES-to-numeracy skill-to-mathematics achievements pathway for Chinese children.

METHOD

Participants

The participants in the present study were sampled from a 15-year longitudinal study of language and literacy (Chinese Communicative Development Inventory, CCDI; Tardif et al., 2008). All the participants (N=309) were born in Beijing and they all studied in public schools in Beijing. Sixty-three children (31 boys and 32 girls) in this study were randomly selected from the large sample based on their mathematics scores in the senior high school entrance examination. The mean age of these students was 15.57 years old (age range: 14.00-16.75, SD = 0.69).

Measures

Parent questionnaire

Two questionnaires were used in the present study. One is the background questionnaire in which parents' educational level and income were investigated. Parents' educational level was measured with a 7-point scale: 1 = *primary grade 3 or below*, 2 = *primary grade 4 to 6*, 3 = *middle school*, 4 = *high school*, 5 = *junior college*, 6 = *university*, 7 = *postgraduate*. Parents' income (monthly income in Chinese renminbi (RMB)) was measured with a 6-point scale: 1 = *less than 300*, 2 = *between 300 and 499*, 3 = *between 500 and 999*, 4 = *between 1,000 and 1,999*, 5 = *between 2,000 and 8,999*, 6 = *more than 9,000*.

Another questionnaire is the behavioral questionnaire in which parents were asked about their children's numeracy performance of using 5-point scale: 1 = *very hard to complete*, 2 = *hard to complete*, 3 = *neither easy nor hard*, 4 = *easy to complete*, 5 = *very easy to complete*. Developed from a theoretical model of early mathematical development (Krajewski & Schneider, 2008), three items were used to describe children's pre-school numerical skill: *counting numbers from one to ten in order (Basic numerical skills), complete addition of less than ten (Linking number words with quantity), name correctly the number of objects <10 (Linking quantity relations with number words)*.

Mathematics achievements test

Mathematics achievements was measured by using a sub-test from the Mathematics Competencies Test Bank (Guo, Cao, Yang, & Liu, 2015) in Grade 7-9, which was designed to measure a student's capacity to formulate, employ and interpret mathematics in four contents including Function, Equations & Inequalities, Geometry, and Statistics & Probability. The sub-test consists of 18 items and students were given 90 minutes, and these items were designed to reflect students' performance in three mathematical capabilities, namely Learning & Understand, Practical Application, and Creation & Innovation.

The four content categories play an important role in Chinese mathematics curriculum, and account for the vast majority of the curriculum standards in middle school. The test measures a student's ability to integrate his or her mathematics knowledge, quantitative reasoning, and calculation skills with solving mathematics problems.

Procedure

Students were tested by several tasks from 1 year-old to 15 year-old. For each student, both of the parents were asked to finish the background questionnaire and behavioral questionnaire when the student was 1 and 5 year-old, respectively. A mathematics test was conducted in students' fifteens, when they have finished their middle school learning.

Statistical procedure

Factor analyses

The aim of the factor analyses was to identify the factor structures of the parents' questionnaire. To make the factor analysis more reliable, based on a larger sample (N=309), an exploratory factor analysis was employed to explore the structure of the seven questionnaire items using SPSS, with a varimax rotation. The factor solution was then tested with a confirmatory factor analysis using Mplus to estimate the factor scores for each participant using the larger sample.

Item response theory

The item response theory analysis was then used to achieve students' Rasch-scaled achievements estimates within a larger sample[1] (N=3,840) with the one-parameter model (Rasch, 1960) and implemented by ConQuest software (Wu, Adams, & Wilson, 1997). The reliabilities (WLE) is .879 for the test, .807 for Function, .745 for Equations & Inequalities, .842 for Geometry, and .867 for Statistics & Probability. Furthermore, the comparison between the present sample (N=63) and the larger sample suggested that the former can stand for the latter (t = .803, p = .422).

[1] This sample is from a Mathematics Competence Test Project for secondary school students in Beijing.

Association analyses

Following the IRT estimating and factor analyses, the association of the estimated person parameter (here the mathematics achievements of each student) and the estimated factor scores were tested using partial correlation and hierarchical regression.

Our hypothesis is that students' early mathematical skill can mediate the effect of parents' characteristics. A mediation model was fitted and a nonparametric approach, bootstrapping test (MacKinnon, Lockwood, & Williams, 2004), was performed to test the indirect effect. The indirect effect was suggested as significant when the bootstrap estimates are different from zero with 95% confidence.

RESULTS

Factor analyses

The exploratory factor analysis indicated three factors with eigenvalues of more than 1. Solution with varimax rotation showed items separately loaded on three factors (see Table 1). Three items about children's performance had high loadings on the first factor, which was referred as early mathematical skill. The father's and mother's educational levels were highly loaded on the second factor, namely parents' educational level. The other two items had high loadings on the rest factor, which named as parents' income.

	Component		
Variables	1	2	3
counting numbers from one to ten in order	**.850**	-.164	.074
complete addition of less than ten	**.655**	-.031	-.291
name correctly the number of objects <10	**.825**	-.047	.173
father's educational level	-.144	**.867**	.163
mother's educational level	-.075	**.907**	.146
father's income	-.011	.056	**.878**
mother's income	.046	.391	**.707**

Note. Factor loadings over .4 are bolded.

Table 1: Rotated factor loadings of exploratory factor analysis with varimax rotation on parents' questionnaire scores.

Correlation and regression

The partial correlations among estimated mathematics achievements and factor scores controlling age and gender were showed in Table 2. Students' mathematics achievements significantly correlated with their early mathematical skill and their parents' educational level (r = .356 and .298, respectively), but not correlated with their parents' income (r = .060), although the educational level and income are highly correlated with each other. Furthermore, students' early mathematical skill is significantly correlated with their parents' educational level (r = .274).

	1	2	3
1. Mathematics achievements	-		
2. Early mathematical skill	.356**	-	
3. Parents' educational level	.298*	.274*	-
4. Parents' income	.060	.004	.609***

Note. *, p < .05; **, p < .01; ***, p < .001.

Table 2: Correlations among mathematics achievements, early mathematical skill, parents' educational level, and parents' income.

Since parents' income wasn't significantly correlate with mathematics achievements, it is not included in the hierarchical regression model. The regression showed that controlling for gender, age and parents' educational level, early mathematical skill uniquely predict students' mathematics achievements (see Table 3), explaining additional 8.1% of the variance in mathematics achievements.

	Variables	β	ΔR^2	ΔF
Model 1	Age	.041		
	Gender	.005	.002	.053
Model 2	Age	.061		
	Gender	-.004		
	Parents' educational level	.299*	.089	5.764*
Model 3	Age	.092		
	Gender	-.001		
	Parents' educational level	.217		
	Early mathematical skill	.298*	.081	5.670*
			R^2 = .172	

Table 3: Hierarchical regression models using gender, age, parents' educational level, and early mathematical skill to predict students' mathematics achievements in high school.

Mediation model

The mediation model was performed in Figure 1. Without the early mathematical skill, parents' educational level can significantly predict children's mathematics achievements. When with early mathematical skill controlled, predictive effect of parents' educational level was not significant. The bootstrapping test estimated a significant 95% confidence interval of indirect effect (indirect effect= .094, 95% CI: [.011, .188]).

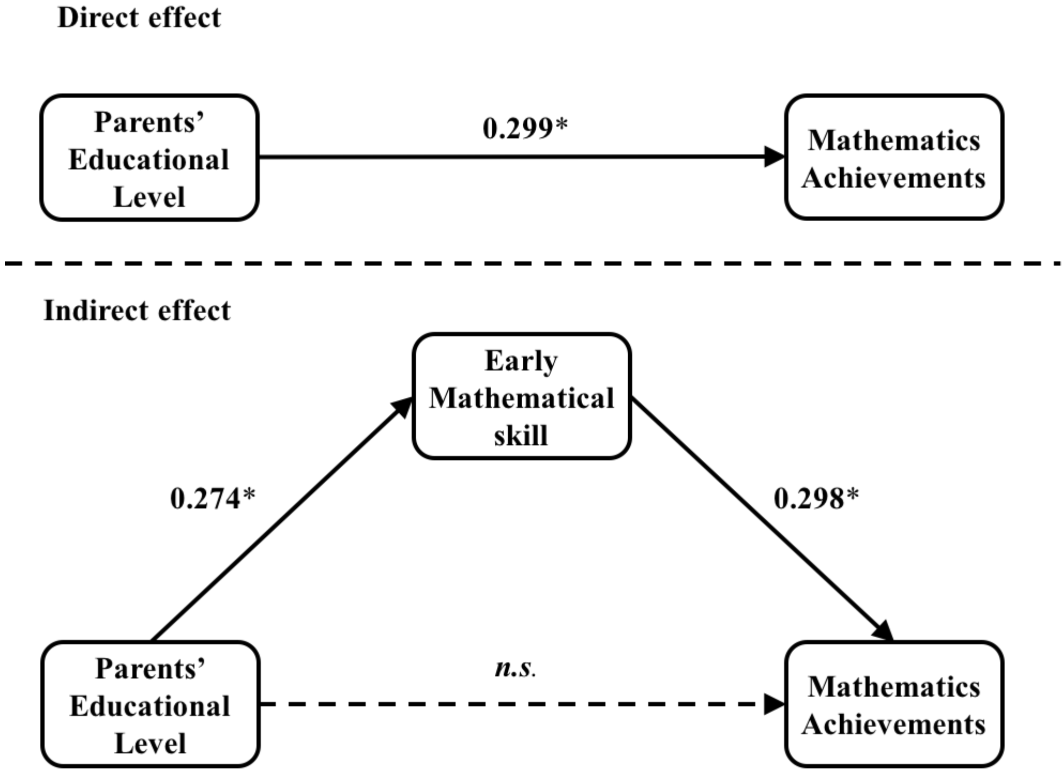

Figure 1. Mediation model with the indirect effect of early mathematical skill

DISCUSSION & CONLUSION

Our results indeed showed that early numerical skill independently predict students' mathematics achievements in high school. Moreover, the effect of parents' education to students' school mathematics was significant and mediated by their numerical skill at 5 years old.

Consistent with previous studies (Davis-Kean, 2005), parents' educational level was proved again as a correlating factor of students' mathematics development. But the income does not have any significant effect on mathematics performance 15 years later, which is rarely revealed in other studies (Sirin, 2005; Van Ewijk & Sleegers, 2010). At least part of the reason may be the operation time, since previous studies mostly investigate SES concurrently with the achievement measure. On the other hand, the Chinese families in low SES even spend a high proportion of home income on their children's education (Wang et al., 2014), so that their family environment caused by income would be comparable with high SES families.

Chinese parents are highly influenced by traditional Confucian values (Li, 2002) and believe in "clumsy birds have to start flying early" and tend to attribute high achievement to hard work, rather than talent (Leung, 2001). Thus, they pay much attention to early childhood education to protect their children from losing at the starting line. The present longitudinal study has produced accumulating evidence for beneficial effect of early numeracy skill on students' mathematics outcomes later. From educational and practical perspectives, early education on numeracy skills is no doubt effective preparation for children before their school learning (Missall & Hojnoski, 2014; Skwarchuk & Smith, 2009). Although family environment, such as parents' educational level, can partly determine children's mathematics achievements, parents achieving low educational level still have chance to help their children to get academic progress by improving their early numeracy skills.

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STUDENT TEACHERS' QUESTIONING BEHAVIOUR WHICH ELICIT CONCEPTUAL EXPLANATION FROM STUDENTS

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Getting students to explain their thinking is one of the big challenges in teachers' work. Previous studies have analysed teacher questioning by focusing on amounts of different types of questions. In this study, I use questioning diagrams to see how questioning develops during the lessons. The data includes video recordings of student teachers' mathematics lessons in secondary and upper secondary school. The data is analysed by constructing questioning diagram for each student teacher and locating conceptual explanations given by students. The lessons which included largest amount of conceptual explanations are further studied. In these lessons the student teachers had lengthy discussions with the students and asked them many kinds of questions.

INTRODUCTION

An essential part of teacher-student interaction is to get students explain their thinking. Explaining is necessary condition for dialogic interaction because ideas need to be shared. In addition, even explaining to one self supports learning because of so called self-explanation effect (Wong, Lawson, & Keeves, 2002). However, there are different kinds of explanation. Kazemi and Stipek (2001) described two classrooms: one where students explained procedures (steps) and one where students explained reasons (why). The teachers in these classes pressed differently for conceptual thinking although some features of teaching were the same.

The two kinds of explanations described by Kazemi and Stipek (2001) correspond to procedural and conceptual knowledge. Procedural knowledge includes procedures which are used to solve problems and conceptual knowledge includes connections between pieces of knowledge (Hiebert & Lefevre, 1986). When explaining reasons, one makes connections. Thus, in this paper, these two kinds of explanations are called procedural and conceptual explanations.

The conceptual and procedural explanations also compares to Toulmin's model (1958). In Toulmin's model a claim (e.g., an answer to a task) is supported by data. Warrant indicates how the claim follows from the data. Thus, procedural explanation describes data for the claim and conceptual explanation gives the warrant.

Teachers can elicit student explanation through questioning. Sahin and Kulm (2008) characterize three kinds of questions: factual questions request a known fact, guiding questions give hints or scaffold solution, and probing questions ask for elaboration, explanation or justification. The first step in getting students to explain is to ask probing questions. However, even though a teacher is asking probing questions it does not mean that students will explain. Franke et al. (2009) found that even follow-up questions did

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 337–344. Szeged, Hungary: PME.

not guarantee explanation. According to their results, the best way to help students give a correct and complete explanation, was asking a probing sequence of specific questions.

In previous studies questioning has been studied by calculating frequencies of questions (e.g., Hähkiöniemi, 2013). This kind of analysis does not consider how questioning develops and progress over time. Lehesvuori, Viiri, Rasku-Puttonen, Moate and Helaakoski (2013) have included this kind of temporal consideration in their analysis by using interaction diagrams which depict the types of teacher talk as a function of time.

This study contributes to studying teacher questioning and student explanation by using questioning diagrams which give more holistic picture of teacher questioning. The aim of this study is to understand what kind of teacher questioning gets students to give conceptual explanations to probing questions. The following research question guided the data analysis: How do student teachers, whose students give conceptual explanations, ask questions?

METHODS

Data collection

The participants of this study consist of 29 Finnish prospective secondary and upper secondary mathematics teachers. The student teachers were in the final phase of the teacher training program. They all had taught several school lessons during the program. The student teachers participated in an inquiry-based mathematics teaching unit taught by the author. The unit included nine 90 minutes group work sessions about the ideas of inquiry-based mathematics teaching. For example, the student teachers practiced how to guide students in hypothetical teaching situations (see, Hähkiöniemi & Leppäaho, 2012). After the unit, each student teacher implemented one inquiry-based mathematics lesson in grades 7–12. All the lessons were structured in the launch, explore, and discuss and summarize phases. During the explore phase students usually worked in pairs or in three person groups. Altogether, there were 16 lessons in secondary school (grades 7–9) and 13 lessons in upper secondary school (grades 10–12). Lesson length was 45 minutes in the secondary school and either 45 or 90 minutes in the upper secondary school. Students used GeoGebra software to solve problems in 17 lessons.

The lessons were videotaped and audio recorded with a wireless microphone attached to the teacher. The video camera and the microphone were synchronized. The hand-held video camera followed the teacher as he or she moved around the classroom. When the teacher discussed with a student pair, the camera was positioned so that students’ notebooks or computer screens could be seen. Although the microphone was attached to the teacher, it captured also students’ talk when the teacher discussed with a group of students. Students’ written notes were collected after each lesson.

Data analysis

Data was analyzed using Atlas.ti video analysis software. All the teachers' questions were coded to probing, guiding, factual, and other questions. The definitions for these codes were constructed on the basis of Sahin and Kulm's (2008) characterizations. The shortened versions of the definitions are as follows:

- Probing questions (code 1): Questions which request students to explain or examine their thinking, solution method or a mathematical idea.
- Guiding questions (code 2): Questions which potentially give students hints or guides solving a problem. Potentially means that students do not have to understand the hint but the questions offers opportunity for this. Probing questions are excluded from this category.
- Factual questions (code 3): Questions that ask for a known fact such as an answer to a task, a definition, or a theorem. Guiding questions are excluded from this category. The difference to a guiding question is that students are not solving a problem and the question does not guide or give hint to solving the problem.
- Other questions (code 4): All other questions such as questions concerning classroom control.

A teacher utterance was considered as a question if it invited the students to give an oral response. For example, utterances such as "explain" were considered as questions even though grammatically they are not questions. On the other hand, grammatical questions were not coded as questions if the teacher did not give the students a possibility to answer the question. Inter-rater reliability for coding probing, guiding, factual, and other questions for a sample of 150 questions was 89 % (Cohen's kappa = .845).

In addition, lessons were coded to launch, explore, and discuss and summarize phases. The episodes when the teacher discussed with a certain student group during the explore phase were marked. After this, questioning diagrams of each student teacher were produced using SPSS and spreadsheet software (see, e.g., Fig. 1). In the diagrams, the horizontal axis shows the time in minutes and vertical axis shows the question type. The beginning and the end of the lesson as well as the lesson phases are indicated by vertical lines. In the exploration phase, the questions asked from a student group (or an individual student) are connected with a line. Questions are marked with red circles or blue triangles so that the symbol changes when the group changes. Questions asked during the launch and discuss and summarize phases are marked with green squares on connected with a line.

After producing the questioning diagrams, students' responses to teachers' probing questions were coded as follows:

- Conceptual explanation: Expresses *why* a result or an intermediate step is achieved using some method, why a property holds or do not hold, or how something represent or means something or how concepts are related
- Procedural explanation: Expresses how a result or step is achieved or how something is done or describes representations
- No explanation

The conceptual explanations were marked with C in the questioning diagrams. Also when a conceptual explanation was identified, it was checked if the teacher had already discussed with this student group and if so these discussions were connected by dashed line.

I looked for lessons in which several student teachers probing questions were answered by conceptual explanations in the explore phase of the lesson. I selected those student teachers whose lessons contained five or more conceptual explanations. I considered these lessons to include high number of conceptual responses because in the other lessons the number of conceptual responses was between 0 and 3. I searched for commonalities and differences in the selected student teachers' questioning diagrams. After this I turned to microanalysis of the video episodes in which conceptual explanations were given.

RESULTS

In four lessons students gave five or more conceptual responses to student teachers probing questions. In these lessons several different students gave the conceptual explanations. The questioning diagrams of these student teachers are given in Figure 1.

Common feature in student teachers 8, 9 and 11 questioning diagrams is that they asked many different kinds of questions from the same students. Thus, based on the diagrams, they engaged in long discussions with the students.

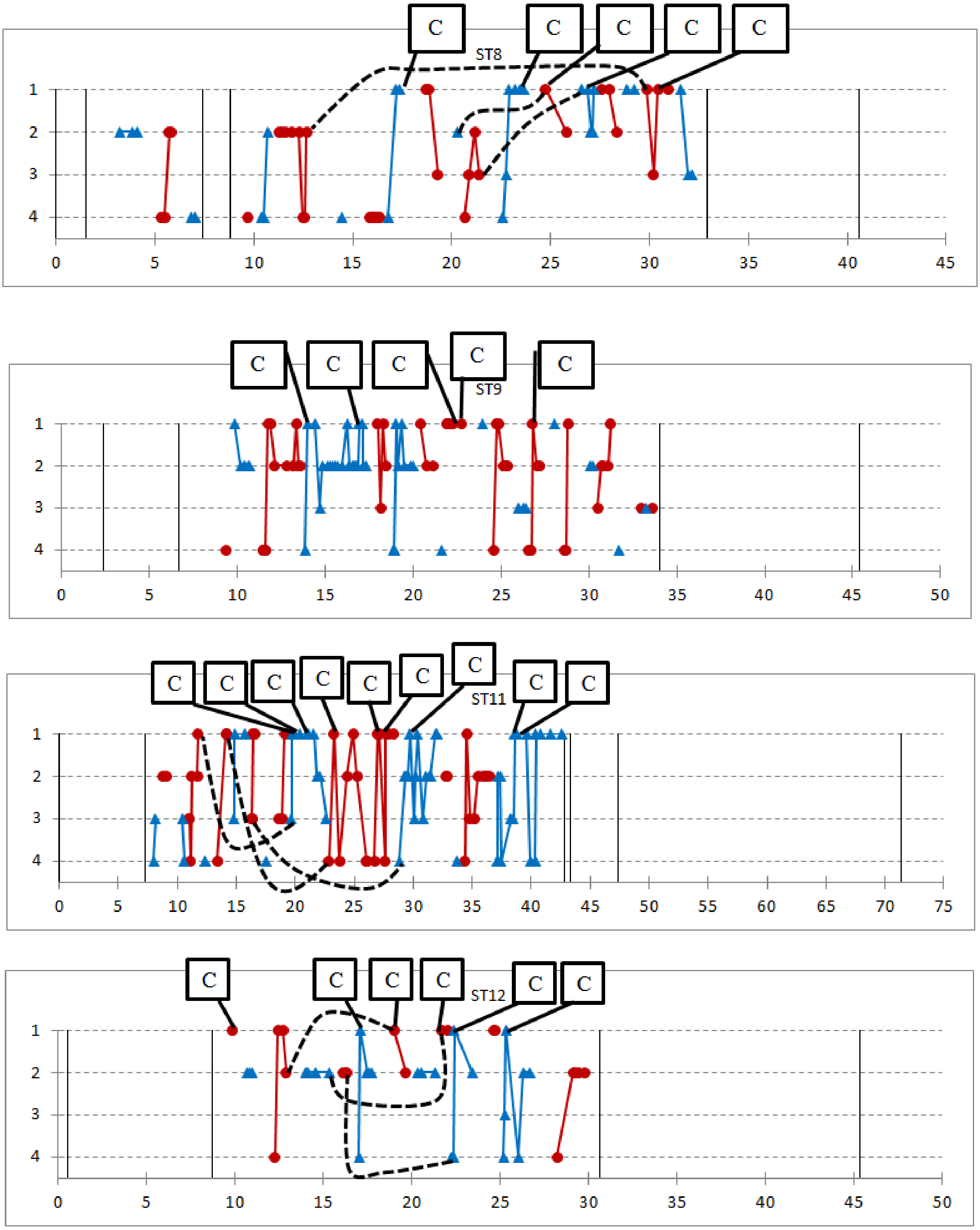

Figure 1: Questioning diagrams of the student teachers whose lessons contained five or more conceptual explanations (1 = probing question, 2 = guiding question, 3 = factual question, 4 = other question, C = conceptual explanation)

The questioning diagram of student teacher 12 seems at first a bit different. However, when we look at how he returned to ask questions from the groups after visiting other groups (dashed lines in Fig. 1), it seems that also he is asking many different questions from the same students, but not just in a row. For example, he first visited a pair of students who were solving how much juice can be made of 1.5 litres of concentrate when 30 % of the juice has to be concentrate:

ST 12: How are you succeeding? [Other question, time 12:15]

Student 1: [Mumbles]

ST 12: Okay, let's see.

Student 2: Is it correct?

ST 12: It isn't quite correct. Let's see. What have you done here? Tell me. Let's see where it goes wrong. [Probing question]

Student 2: Uhm. I don't know. We thought that 30 %, it has to be multiplied by 7. [Procedural explanation]

ST 12: Why it has to be multiplied by 7? [Probing question]

Student 2: I don't know.

Student 1: I would have understood, that I think that you multiply by 0.70. [Procedural explanation]

The students had solved the task as shown in the crossed part of figure 2. The student teacher asked why the students had multiplied by seven and thus started to probe reasons. Then the student teacher guides students to use *x* and lefts the student to continue. Later he comes back to this group:

ST 12: Explain a little what you have done here. [Probing question, time 18:58]

Student: We took first 10 % which is this 0.5. Then we multiplied it by 7 to get 70 %. Then we added the 30 % to 70 %. [Conceptual explanation]

The students were still not using *x* but now they gave a conceptual explanation. The explanation is conceptual because in addition to describing the steps, the student also indicates that 0.5 is multiplied by 7 to get 70 % in this case.

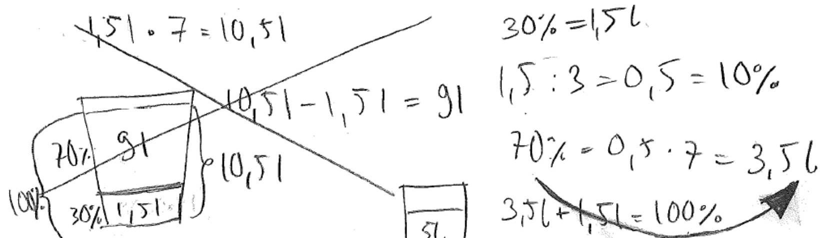

Figure 2: Students' solution of how much juice can be made of 1.5 litres of concentrate when 30 % of the juice has to be concentrate.

Also two other student teachers returned to a previously visited group when they got conceptual explanations (see Fig. 1.). Only student teacher 9 did not visit the groups which gave conceptual explanation before.

In many cases the probing question which yielded a conceptual response was not the first question or the first probing question. For example, student teacher 8 asked questions from a group who had drawn a line representing a situation in which entrance fee is 2 euros and time based fee is 5 €/h:

ST 8:	What is the meaning of your line? [Probing question, time 16:48]
Student:	It is the second task. [No explanation]
ST 8:	Okay. Yeah. On what grounds did you think that it would be like that? [Probing question]
Student:	Because here are euros and here is time and always when it plays an hour it is 5 €. And here it has played 2 h, then it is 10 €. [Conceptual explanation]

The students did respond properly to the first probing question. When the teacher reworded the question, the students gave a conceptual explanation. Also in the other lessons, which contained fewer conceptual responses, the conceptual explanations were often given when the teacher asked many different types of questions or when the teacher focused the probing question based on the student's response.

DISCUSSION

The results of this study show in what kind of conditions it is possible to get the students to give conceptual responses to probing questions. One of these conditions is that the student teachers engage in lengthy discussion with the students and asks several different types of questions. These kinds of discussion can be regarded as more authentic than short discussions following initiation-response-evaluation pattern (Mehan, 1979; cf. Lehesvuori & al., 2013).

Another feature which is connected to getting the students to give conceptual questions, is asking several probing questions in a row so that when students give non-conceptual response, student teacher modifies the question based on students' responses. Similarly Franke et al. (2009) noticed that probing sequence of specific questions was most efficient way to get the students to give a correct and complete explanation. Thus, the results of this study support Franke et al.'s (2009) findings.

Also returning to ask questions from the same students was used when students gave conceptual explanations. Keeping track of all the different paths taken by the students, supporting and even relating them to each other is one of the big challenges in orchestrating students' problems solving (Stein, Engle, Smith, & Hughes, 2008). When a teacher manages to keep track and return to continue the discussion, the students are perhaps in better position to express their idea as they had time to think.

The questioning diagrams were useful research tool as they made it possible to compare student teachers' questioning more holistically and notice commonalities and differences. In future research, the questioning diagrams could be used to study how teachers' questioning habits change over time.

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IMPACTS OF STUDENTS' DIFFICULTIES IN CONSTRUCTING GEOMETRIC CONCEPTS ON THEIR PROOF'S UNDERSTANDING AND PROVING PROCESSES

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This RR is part of a comprehensive study whose goal it is to investigate the effects of the process of constructing geometric concepts on students' proving processes related to these concepts. In the current RR we focus on the effects of visual difficulties in constructing concepts on proving processes. We found three effects: the impact of the difficulty to identify a non-prototypical example, the impact of the failure to identify a common element of two shapes, and the less strong effect of using self-attributes of a single drawing.

INTRODUCTON AND BACKGROUND

The study reported here is part of a larger research. The larger research has three goals, namely to investigate 1. the effect of visualization on students' construction of geometrical concepts and their definitions, 2. the effect of visualization on students' ability to prove in geometry, and 3. the effect of definition understanding on students' ability to prove in geometry. The current research report focuses on goal no. 2.

In the mathematics education research literature one can find many studies that focus on the construction of geometric concepts, and specifically on the mutual interaction between their concept definition and students' concept images (e.g., Fischbein, 1993; Fujita & Jones, 2007; Hershkowitz, 1987; Vinner & Hershkowitz, 1980). One of the main findings of the research literature is the prototype phenomenon: Vinner & Hershkowitz (1980) found that, for each geometric concept, there is at least one prototypical example. The prototypical examples are usually acquired first, and are therefore prominent in the concept image of most learners. Prototypical examples are usually the examples of the concept with the most attributes - the critical attributes of the concept and self-attributes that are not critical. Often, these non-critical attributes have dominant visual properties, which have an effect on the concept's identification, classification and construction. A related difficulty is interwoven in the mediation of the geometrical objects by graphical representations - their drawings. Parzysz (1988) pointed out that the drawing is unique, wherever it represents a set of objects, which is usually infinite and has common critical attributes. Laborde (2005) showed that students often regard the unique and particular drawing on paper as the object itself, rather than the abstract object represented by the drawing. Thus drawings may cause difficulties for students in their proving processes while using the particular attribute of the single drawing instead of using the critical attribute of the geometric concept.

Other studies focus on the difficulties of proof construction, difficulties of understanding proofs, understanding its essence and understanding the need for it (e.g. Martin & McCrone, 2003). However, we could not find in the literature a clear focusing on the relationships between these two research domains - the construction of geometric concepts and the proving processes related to these concepts. The present research attempts to fill the gap. The goal of the current study is to investigate possible relationships between these two areas of learning geometry, more specifically to investigate the effects of students' processes of constructing geometric concepts, including the mutual interconnections between concept images and definitions, on students' proving processes related to these concepts.

METHODOLOGY

Population: The participants are 90 students from a regional high school in an Arab community in the centre of Israel; they learn geometry with three different teachers in three parallel classes, which are considered to be at the highest mathematical level among the seven parallel classes in this school. The teachers have a first degree in mathematics from the universities in the country and each has more than ten years of experience in teaching mathematics.

Research tools: The main research tools of the large research include three questionnaires, one for each goal of the research. The questionnaires were distributed at time intervals sufficient for analysing the results of each questionnaire and use its findings in the design of semi-structured interviews with about 10% of the participants, as well as in the design of the next questionnaire. The second questionnaire, the one used for the current study, deals with the effect of visualization in concepts constructing processes on proving processes.

In some questionnaire tasks, the participating students are asked to prove or to reflect on imaginary students' proofs. During such a reflection, students have opportunities to use *critical thinking*; they test the proof made by the imaginary student. Detailed analyses of the questionnaire tasks and of students' responses are given in the next section.

DATA COLLECTION AND FINDINGS

The data of the current questionnaire were collected while the participants were in grade 11. The questionnaire includes 4 tasks and was administered at the end of the first semester.

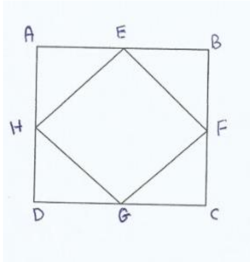
The First Task (Figure 1):

The claim and the proof of Salim are wrong (see Figure 1 below); he uses the attributes of the quadrilateral example in the drawing (a square) instead of using the critical attributes of a general quadrilateral.

ABCD is a quadrilateral, E is the midpoint of AB, G the midpoint of DC, F the midpoint of BC and H the midpoint of BD.

What is the type of the quadrilateral HEFG? Prove your claim!

Salim drew the following shape, he claimed that EFGH is a square and wrote the following proof: " E, F, G, H are the midpoints of sides of the square ABCD, therefor AE = EB = BF = CF = CG = DG = HD = AH.



So the triangles $\Delta AHE, \Delta BEF, \Delta FCG$ and ΔHDG are isosceles right triangles and all of them are congruent.

Therefore EF = FG = HG = HE. Thus the quadrilateral EFGH has four equal sides! In addition, because $\angle BEF = \angle AEH = 45^0$ (the triangles ΔBEF and ΔAEH are isosceles) this lead to the result that $\angle HEF = 90^0$. To conclude: the quadrilateral $EFGH$ has four equal sides and a right angle, so it is a square!

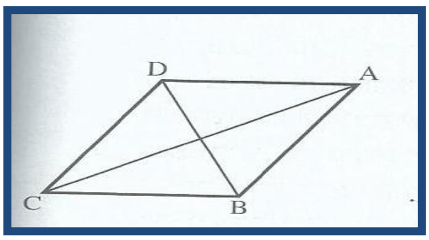
Are Salim's claim and proof correct? Explain your answers!

Figure 1: The First Task.

The findings point to the impact of the use of the attributes of a "single drawing" instead of the critical attributes of the given figure in the proof. As a result, a third (37%) of the 90 pupils have concluded and accepted a wrong proof, they "fell into the trap" and claimed that Salim's proof was correct. In other words, these students went astray after the drawing of Salim (a square) instead of dealing with the data expressed in the written task. 82% of the students who claimed that Salim's proof is correct referred in their explanations only to the square and gave right proof for square, wrong proof for square or missing proof to the square. Approximately 56% of the students claimed that Salim's proof was wrong. Most (76%) of them justified their claim on the grounds that Salim used the wrong information; this means that they noticed that the drawing of a square does not match the data of the problem and therefore Salim's claim and proof are incorrect.

The Second Task (Figure 2):

ABCD is a rhombus in which the length of the diagonal AC is twice the length of its altitude.



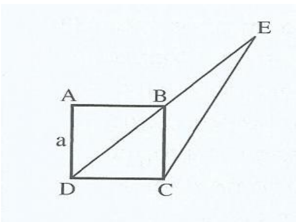
Prove that $\angle ACB = 30^0$

Figure 2: The Second Task.

Constructing an external altitude (from vertex A to BC), creates a right angle triangle in which the length of the perpendicular side is half of the hypotenuse length. The failure to identify, or the inability to construct non-prototypical altitude (external altitude) in a triangle or quadrilateral, prevented students to start proving. About two-thirds (63%) of the students failed to prove the claim because they didn't succeed to construct an external altitude. These students drew an internal altitude or no altitude; among these students, 83% used unjustified or false assumptions (Dvora & Dreyfus, 2014) or made other mistakes in the proof. Only about 37% of the population were able to identify the external altitude from vertex A; among these students, 82% proved the claim correctly. In addition, all the students who wrote a correct proof based their proof on constructing the needed external altitude. The strong visual properties of the prototypical example of the altitude concept (internal segments) affect the students' ability to prove. In the interviews we have the opportunity to explore whether only the construction of the external altitude causes the inability to prove. When we helped the students to construct the external altitude in the interviews, they immediately came to the right proof. This behaviour strengthens the conclusion that the inability to identify the external altitude prevents students to start proving.

The Third Task (Figure 3):

ABCD is a square in which we extend the diagonal BD so that BD=BE.



a) Are the triangles $\triangle BDC$ & $\triangle CBE$ *congruent?*

b) Are the triangles $\triangle BDC$ & $\triangle CBE$ *having an equal area?*

Figure 3: The Third Task.

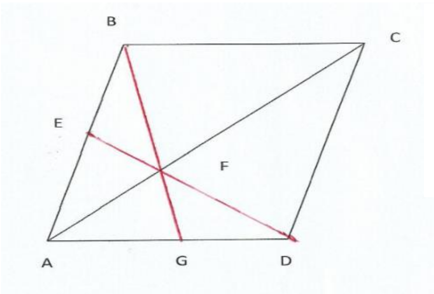
25% of the students gave a wrong answer for the first question and claimed that the triangles are congruent; in order to explain their responses they wrote a wrong proof for their claim. About 70% of the students answered correctly the first question and claimed that the triangles are not congruent. The data show that more than the half (60%) of the students who claim that the triangles are non-congruent indicated that areas of these triangles are not equal. The majority of the student's explanations who indicated non-equality of the areas of the triangles (87%) were based on the wrong assumption that for an equal area the triangles have to be congruent. E.g. one student wrote: *"all the sides are not equal"* or another student wrote: "*all the angles are not equal"*.

Only about 30% among the students who claimed that the triangles are not congruent also indicated that the areas of the triangles are equal. Approximately 58% of these students (who noted that the triangles have an equal area), explained their responses by using the claim that the triangles have an equal bases and a common altitude. About

26% among these students who noted that the triangles have equal areas explained their responses by using a trigonometry.

The Forth Task (Figure 4):

$ABCD$ is a parallelogram, G is on the side AD. Given: AG=4cm, GD=3cm. The area of the triangle ΔAGF is equal to $5cm^2$



Is it possible to calculate the area of the triangle ΔGDF ?
If not explain what is missing! If it is possible do it!

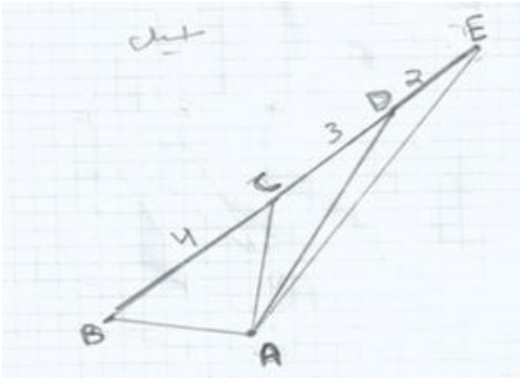
Figure 4: The Forth Task.

In the fourth task the students are asked to calculate the area of one triangle; this calculation is based on the area of another triangle and on having a common altitude for both triangles.

For the first question whether it's possible to calculate the area of the triangle ΔGDF we get that the majority of the students (57%) claimed that it's impossible to calculate the area of the triangle ΔGDF . 44% among them didn't identify the common altitude of the triangles $\Delta AGF, \Delta GDF$ and indicated that the altitude is missing. For example one student's response was "*No, it's impossible to calculate the area of the triangle ΔGDF because no altitude is given*". 44% among the students who claimed that it's impossible to calculate the area of the triangle ΔGDF indicated that that there is missing information like angles or sides.

For the second question, 75% among the students didn't succeed to calculate the area of the triangle correctly. The data show that the difficulty to identify or to construct the common altitude of the triangles prevented many students to calculate the area of the triangle. It should be noted that the common altitude of the triangles is an external altitude for one of the triangles. The question which arose was: whether the inability to identify the altitude arises from the fact that it is an external altitude to one of the triangles, or whether it is due to the inability to identify a common altitude even if it is an internal altitude. The interviews allowed us to take another step forward in order to answer this question. Here is an episode from one of the interviews, in which, in order to calculate the ratio between the areas of two triangles, one has to identify their common altitude which is internal to both triangles:

Episode (I – interviewer; A – Aseel, a student), regarding the following task.



1	I:	Can you calculate the ratio between the areas of the triangles ABC and ABD?
2	A:	The ratio?
3	I:	Yes the ratio between the areas of the triangles?
4	A:	Maybe 4:7.
5	I:	4:7? Why?
6	A:	This BC equals 4 and BD equals 7.
7	I:	So, why you conclude that the ratio is 4:7?
8	A:	Because BC equals 4 and BD equals 7.
9	I:	What is the connection? How can you calculate the area?
10	A:	By multiplying the length of one side and the length of the altitude to it.
11	I:	Where is the altitude?
12	A:	(silent).

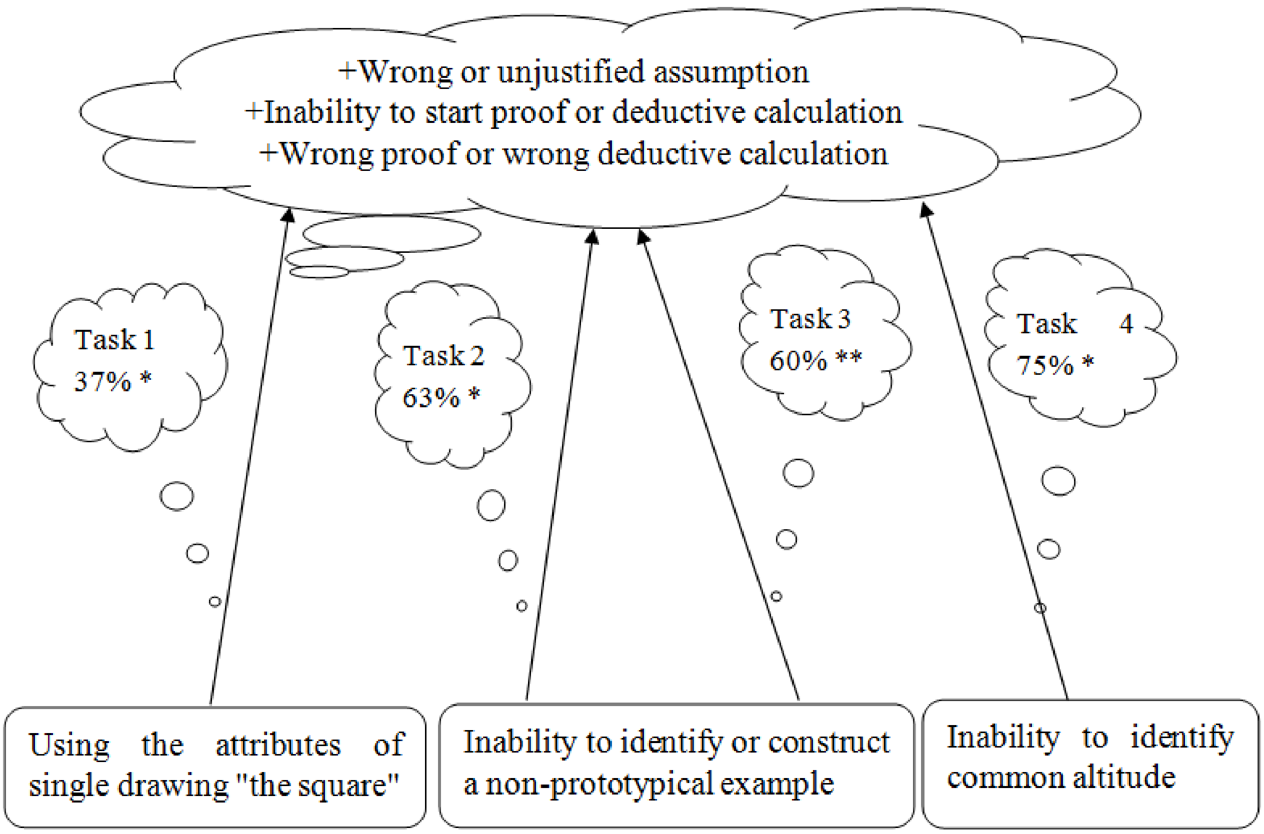
Aseel knew how to calculate the ratio between the areas of the triangles, but she didn’t know to justify her response. Aseel didn't identify the common internal altitude of the two triangles. When we tried to help and push Aseel to reach the correct argument that there is a common altitude for both triangles (line 11 in the episode), Aseel continued not to identify the common internal altitude. All of the nine interviewees except one, who were asked to calculate the ratio between the areas of the two triangles, reacted like Aseel.

CONCLUDING REMARKS

The goal of the current study was the investigation of the influence of visual factors associated with geometric concept construction on proving processes. The findings show the impact of three effects on the ability to prove: i) using self-attributes of the "single drawing" instead of the critical attributes of the figure in the proof task usually led to wrong assumptions and from there to wrong proofs (Task 1). ii) Failure to identify or inability to construct non-prototypical examples, such as an external altitude in a triangle or quadrilateral, or non-congruent equal area shapes, usually prevented students from starting the proving process, or led to wrong assumption (Tasks 2&3).

iii) Failure to identify a common element to two geometric shapes, such as a common altitude of two triangles, usually limited the ability of students to prove or caused them to make wrong assumptions (Task 4, Episode from interview).

Figure 5 summarizes the main findings of the study. The impact of the three effects are not equal as shown on the arrows by the percentages of the population who were not able to make transition from visual representation to justification. In the population of the current study, the failure to identify or inability to construct non-prototypical examples, and the impact of the failure to identify a common element to two geometric shapes, are far greater than the impact of the use of an attribute of a "single drawing" on students' proving processes. The interviews confirmed, sharpened and highlighted the findings from the questionnaire, mainly about the failure to identify a common element for two geometric shapes and about the prototype assumption that the congruence between shapes is a necessary condition for the equality of the shapes' areas.



* Percentage of the entire research population

** Percentage of the students who claim the triangles are not congruent

Figure 4: The main findings

In summary, the uniqueness of this work is that it considers geometric concept construction and the development of the ability to prove as a sequence of closely connected abilities. The research findings indeed indicate that the zones of concept construction and proving in geometry are a continuum on which the geometric concepts construction process and the difficulties associated with them have clear effects on the students' proofs processes and their ability to prove.

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SILENT GAZING DURING GEOMETRY PROBLEM SOLVING, INSIGHTS FROM EYE TRACKING

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The authors examine data on student gaze from their different research perspectives as they consider the contribution such data collection methods can make to study of student mathematics learning within the complexity of classroom activity. Findings from study of eye tracking, and body language are compared to identify evidence of learning of two middle school students in Finland. The gaze of one student is tracked using a head-mounted eye-tracking device. Body language was examined independently before findings are compared, to identify what evidence each does offer. It was found that body language and eye tracking employed together were insufficient to provide information about the thinking students undertake even though eye tracking provides information about their focus of attention, and body language indicated intervals of engagement consistent with eye-tracking findings.

INTRODUCTION

When visual information (e.g. diagrams) are presented in the classroom, an important part of student behaviour is the quiet processing of that information. The student looks at the diagram and its different parts, trying to make sense of it. How a student looks at the diagram is influenced by their preliminary intuitions and the ideas activated by the task context (Knoblich, Ohlsson, & Raney, 2001). Tracking student visual searching of ideas informs us of their problem solving behaviour and how this search is influenced by the learning context. In our article, we are exploring the possibility of using eye tracking to study such quiet gazing in the context of non-routine problem solving in an authentic classroom context.

Communication consists of more than just words and diagrams; gestures, glances, body movement, voice articulation, and prosody are also important aspects of it (Arzarello, Paola, Robutti, & Sabena, 2009; Mercer, Wegerif, & Dawes, 1999; Radford, 2009; Roth, 2012). In this article, we are analysing student behaviour through their verbal utterances, their body language, and the direction of their gaze. In relation to the present study, body language is one of the lenses employed to study student engagement during the lesson. By employing both eye tracking, and body language lenses, it is intended to find out more about how eye tracking can contribute to research designs: providing a new way to study student learning within the complexity of classroom learning. The inclusion of body language in this study is intended to examine whether eye tracking offers additional information that is not apparent through lesson observation.

The Quality of Experience Framework **EyDUPLEx** (see Williams, 2005) was developed using Csikszentmihalyi and Csikszentmihalyi's (1992) descriptions of 'flow', a state of high-level engagement during creative activity. It was employed to

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 353–360. Szeged, Hungary: PME.

identify possible intervals of flow (Williams, 2010). Where a student displayed most of the body language indicators, further analysis was undertaken to see whether creative development of mathematical ideas was occurring. In our study, we are working on the premise that students displaying few but some body language indicators associated with high-level engagement are likely to be more engaged than students displaying none of these indicators.

We use eye tracking to examine student's visual attention during problem solving, specifically during silent gazing. There is quite extensive eye-tracking research on how visual information is being processed. Experts are known to find the task-relevant features of the visual information faster than novices and their visual attention is focused more on the relevant areas of the visual stimulus (Gegenfurtner, Lehtinen, & Säljö, 2011). Regarding insight problems, Knoblich et al. (2001) add that even novices are more likely to solve a task successfully, if they attend to the relevant areas. Moreover, they observed that when people are stuck, they tend to stare at the problem. So far, eye-tracking studies are done almost exclusively in laboratory settings lacking the authenticity of a real classroom. Moreover, most eye-tracking studies focus on controlled experiments, whereby more complex behaviour included in non-routine problem solving is seldom addressed.

In class, a student is an active agent, whose behaviour is determined by his or her needs, goals, identity, and resilience. At the same time, for many students, their behaviour is largely reactive to changes in the environment, especially to what the teacher and the student's peers do. Due to such complexities within classroom interactions, no research method has been identified that provides a complete account of meanings of this behaviour and reasons for it. Clinical interviews and think-aloud protocols can inhibit thought processes and social interaction in class, and thus limit ecological validity. Reconstructive post-lesson interviews in which general questions are asked may not access memory traces and thus not be valid. Where stimuli – such as video-stimulated recall – are included with the interview, the likelihood that students access memory traces is increased and thus the validity of the data (Ericsson and Simon, 1980). Observations of facial expressions, brain imaging, and other physiological measures contribute additional information but fail to capture the meanings students associate with their behaviour. Yet, each new methodology has shed light on some new aspects of the complexity of student cognition.

In the current study, our research question is: "What kind of *unique* information does eye tracking give on student problem solving behaviour during quiet visual processing, that is not accessible from a careful analysis on their video recorded body language alone?"

METHODS

Participants and apparatus

The data is obtained from a Finnish ninth grade class in a school in Helsinki that has a well-regarded academic reputation. The subject wearing the glass frame (Kimi) and a peer working with him (Saku) volunteered to participate in the study. We had two ordinary video cameras in the class. One video was recording the class from behind, focused on the teacher and the board. A second video focussed on Kimi and Saku.

Figure 1. The eye-tracking device (model used, not the study participant).

In order for the researchers to monitor Kimi's visual attention, he wore a head mounted eye-tracking device. This device was developed in collaboration between the Finnish Institute of Occupational Health and Helsinki Institute of Information Technology (Lukander, Jagadeesan, Chi, & Müller, 2013). The device consists of a glass frame equipped with two miniature cameras (see Figure 1); one camera follows the eye while the other camera points to the viewing direction. The prototype of that device has been used in the current study. The associated software computes the direction of gaze, producing a video scene with a marker indicating the locus of visual attention of the person wearing the glass frame. The eye-tracking glasses are connected to a laptop with two cords, which prevent the student from moving around but do not restrict movement while seated. The used tracking scheme is robust against small changes in lighting conditions and especially against movement of the glasses during a measurement.

Procedure

The teacher opened the lesson with the problem-solving task designed by the researcher (Hannula). The students worked in pairs, each pair sharing a tablet and using GeoGebra software. As an introductory task, the teacher guided her class to solve a problem, where they needed to find three lines, forming a triangle, which contained three given points, and which had the least possible area. The purpose of this task was to highlight the method of modifying the triangle, especially when using GeoGebra software on a tablet. The students were then asked to solve a similar, but more complex problem with four given points: A=(0,1), B=(0,5), C=(3,1), and D=(5,2). Again, they were asked to find three lines, forming a triangle which contained all four points, and which had the least possible area. The task is a non-routine problem for students.

Data analysis

Our data analysis is twofold. First, we analysed student body language and then looked at what additional detail we could obtain from the eye-tracking data. Body language was examined using the target student video, initially by Williams (who does not understand Finnish). Thus, initial analysis had limited influences from knowledge of content of the interactions (other than teacher's diagrams and gestures). The indicators for the quality of experience framework **EyDUPLEx** were formulated using the flow construct (Csikszentmihalyi & Csikszentmihalyi, 1992). Flow is a state of *high positive* *affect* during *creative activity*. During flow, people *lose all sense of time, self, and the world around*, because all of their *energies are focused on the task* at hand. The set of body language indicators are Ey (eyes on the task or person working with the task), D (body directed towards the task, including direction of knees, leaning over, hand on the paper while another is working with it), U (unaware of the world around: focused intently on the task, not responding to what is happening around them, often motionless if looking at something and thinking about it), P (participating in the task, orally, and or physically), L (latching to the ideas of others by completing or expanding on the statement of another), and Ex (exclamations of surprise or pleasure).

The target of student gaze was calculated based on data from the two cameras on the eye-tracking device. As an output, a video was produced with a marker for the target of student gaze. A different computation was required for each viewing distance, due to the parallax effect. Hence, we produced two videos: one for a short viewing distance (to own desk) and another for a longer viewing distance (to the board). Occasionally, the gaze was outside the visual field of the video, mostly when the student looked at the tablet. As the raw data included the coordinates of the gaze direction, we could determine the general gaze target, when only a little outside the video's visual field. When interpreting eye-tracking data, it is important to distinguish between foveal and peripheral perception because it is only in the fovea (spanning less than 2 degrees of visual field), that we can identify finer structures such symbols and fine articulation of gestures (Gullberg & Holmqvst, 1999). For peripheral perception, light and motion recognition is good but textual recognition is poor. On this basis, we assumed that the student would notice only those finer details on screen within a 10-centimetre radius from the marked gaze location. Moreover, we are aware that even when the student gaze is on a target, it does not guarantee that the student attends to the target. Hence, instead of individual glances, we looked at the sequences of gazes and interpreted them in the overall context of student problem solving.

RESULTS

We begin the description of events, from when the teacher set a new task requiring students to find the triangle with the smallest area that encloses four given points. She projected a large triangle that enclosed all four points on the screen, and suggested student pairs try to find a triangle that fits the constraints but has a smaller area. The two focus students worked together on the tablet, Saku constructing and modifying the triangle and Kimi using the area tool to display the area. Their (slightly imprecise) solution can be seen in Figure 2. They spontaneously shared their area result with other pairs of boys, making clear that the area they had found was smaller than the areas found by others. The teacher presented this solution on board and encouraged the class to try to find other possibilities (better alternatives). Soon after, Saku left his seat to find how another group (with a larger area) had positioned their rays (away approximately 30 seconds), Kimi picked up the tablet, inspected it from several angles and then appeared to use it. On return, Saku stated: ”let's try one” and took the tablet from Kimi (who gave no indication of objecting). Kimi contributed the pair's answer to the teacher as the smallest area found [time: 37:47].

Some of the interactions [after 37:47] are now provided and interpreted employing both perspectives. For most of the time, Kimi maintained his slouched position and shifted his gaze multiple times from board, to teacher, to tablet and Saku.

At 37:48, the teacher stated: "two rays [go] through A and C and B and D [pointing at the lines as she spoke]. Did anyone have a different?" Kimi focused his attention on the triangle, turning to the tablet when Saku responded to the teacher's request for another possible triangle. Kimi examined the solution on the board [Figure 2], then looked 'out into space' in a slightly downward direction [38:20-38:24] which could have been staring into space, reflecting, developing new mathematical ideas, or engaging in non-task related thought. He then paid close attention to the teacher actions at the board as the teacher showed Saku's solution "it was like this [moved rays in Figure 2 to make Figure 3] one of the rays went through those C and D", and encouraged students to search for other possibilities [38:24-38:40]. Kimi then glanced at the tablet and asked Saku what he was doing [38:41]. Saku presented a hypothesis to the teacher "(So that) is this we should at how long their shared distance (always)? Is that right? [38:42]. At 38:45 when the teacher discussed the second nother solution [see Figure 3], Kimi focused on the board, the teacher, then the GeoGebra tool on the screen. The teacher continued to encourage students to search for other possibilities, Saku operated the tablet, and Kimi looked at the tablet screen before the teacher pointed at the ray through D and C: "Could this ray's direction yet be changed?" [39:06].

Both students turned their gazes to the board and Saku asked: "Uh, which one?" She traced the line with her finger from D to E: "This ray, could its direction be changed?" [39:12]. Over the next 7 seconds Kimi remained motionless. His eye-tracking data indicated attentive examination of the triangle. First his gaze followed the movement of teacher's finger with short delay, then he glanced at the teacher's face, returned to diagram, and his gaze again traced the ray from D to C, paused at C, continued on for half a unit towards E and then traced back to point C. This segment of trace of Kimi's gaze is represented by the jagged curve with an arrow at the end [Figure 3]. The eye tracking, in conjunction with body language could indicate less awareness of the world around. Such focused attention, and motionless body has been associated with student engagement with spontaneously focused questions (Williams, 2005).

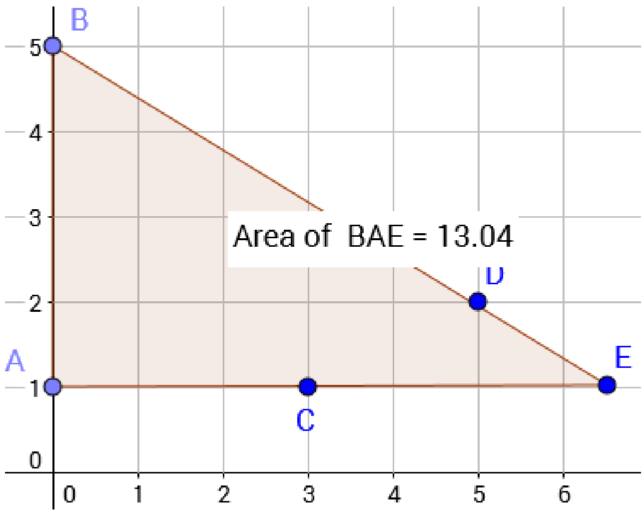

Figure 2. Kimi and Saku's first solution.

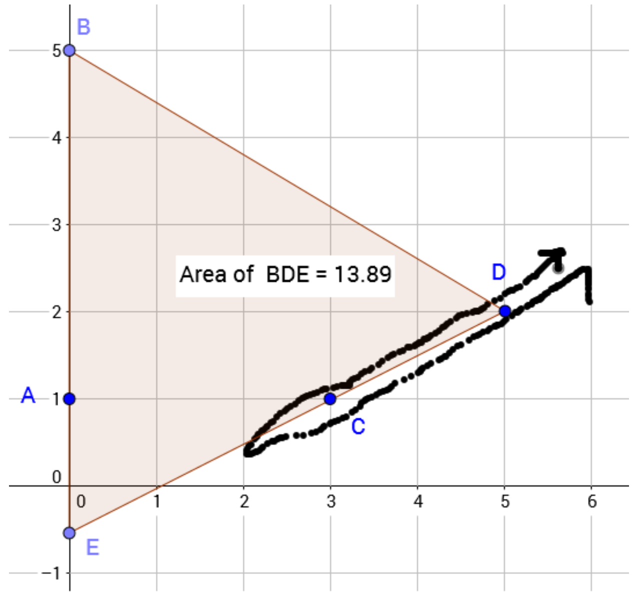

Figure 3. Kimi's gaze at 39:14

After a brief glance at the bottom right corner of the screen, Kimi returned to explore the diagram, gazing along a horizontal track near the point C as shown in Figure 4. Again, his gaze went down and returned to explore the diagram (Figure 5). His gaze traced segments of the line DE and horizontally around the point C [39:12-39:19]. In this movement of gaze, there is some indication of diverging from the two found solutions and exploring a possible third line that is not horizontal nor the line DE.

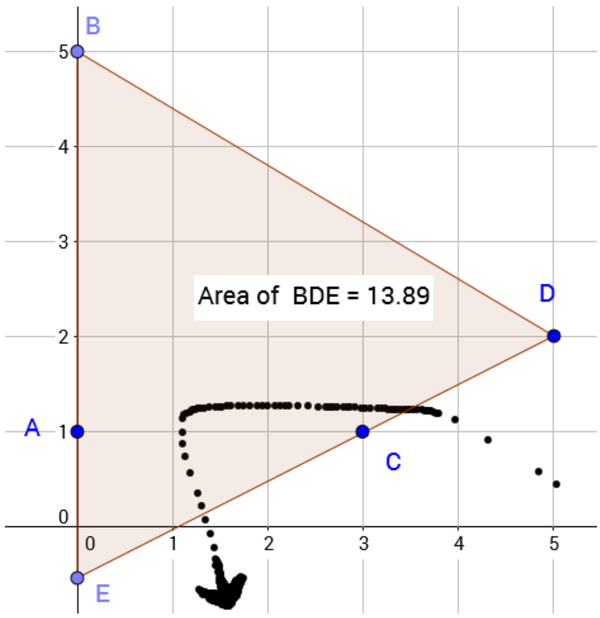

Figure 4. Kimi's gaze at 39:15

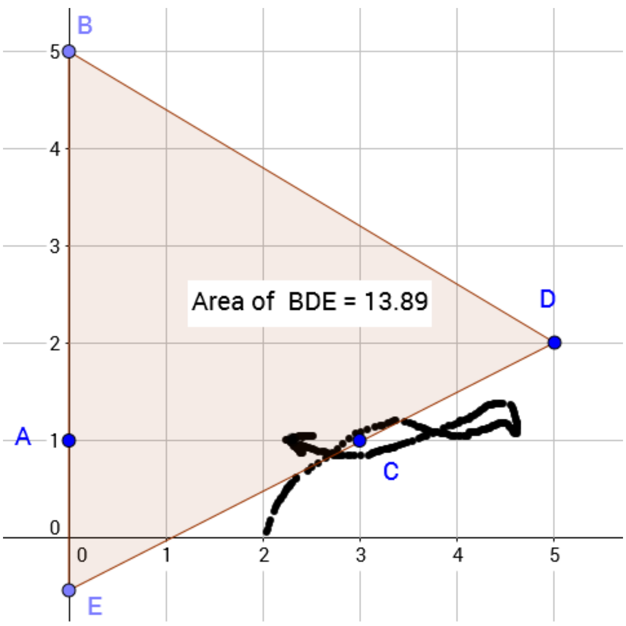

Figure 5. Kimi's gaze at 39:17

When the teacher said: “These were quite good these both solutions 13.86 and 13.29!” [39:19], Kimi’s gaze turned to the teacher’s face and his explorative tracing on the triangle ended. He did return to gaze at the information on the screen, but there was no indication of any systematic attention on any lines of the diagram even when the teacher asked again: “Would any ray have a steepness something between the two?”

Finally, the teacher led the class to find an intermediate version: "Well, let’s try it. I suggested that if (it went) somewhere there somewhere there middle, middle ground so could we get it better [40:32].” The teacher manipulated the triangle moved the angle D a little out and then gradually dragged the point E up, asking the students to say when to stop. Kimi displayed lip movement just before he growled “*Nnoow*” as though waiting for the ray to reach a certain position before he spoke [40:49].

DISCUSSION AND CONCLUSIONS

In this case, body language did not help to identify learning not associated with flow when it was employed as the sole theoretical lens. As there was not much verbal or bodily action, it was unclear from his body language alone whether Kimi was cognitively engaged. Whether this is always the case requires further study. What is interesting in this episode is the ambiguity of Kimi’s engagement based on an analysis of his body language, yet the increased interpretation of body language that was possible when informed by the eye-tracking data. Body language showed the direction of his gaze, when he participated orally or physically in the task. It did not provide information about whether he gazed in an unfocused way or specifically focused his attention. With eye tracker, we gain additional information about the target of his attention and based on this, we obtain a better picture of his problem solving behaviour during the silent gazing, especially the three consecutive gazes (figures 3, 4, and 5) exploring the existing and alternative solutions. Even though we had additional information from the eye tracker, we did not know what mathematical thinking Kimi undertook as he focused on particular diagram features. Body language and eye tracking employed together were insufficient to provide information about the *thinking* students undertake even though eye tracking provides information about their focus of attention, and body language indicates intervals of engagement consistent with eye-tracking findings. A potentially fruitful area for future research would be to include post-lesson video-stimulated interviews that include eye-tracking video as part of video stimulation, to hopefully enrich student reconstruction of students’ mathematical thinking by providing additional memory traces.

Our study shows that eye tracking can be employed as an additional tool to gain further insights into classroom learning. As technologies improve and eventually become cheaper the headpiece will hopefully become less intrusive, and more than one student’s gaze will be able to be tracked during the one lesson.

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ADDRESSING STUDENTS' DIFFICULTIES IN EQUIVALENT FRACTIONS

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215 8-10 year olds undertook exploratory tasks devoted to addressing common fraction errors on an intelligent learning platform. Students continued to make errors but there was improvement in their overall performance. We make the case that partitioning of virtual representations while concurrently showing a changing fraction symbol was instrumental for some students addressing their difficulties with whole number bias and the shift from additive reasoning towards multiplicative reasoning.

INTRODUCTION

Fractions are widely-acknowledged as a challenging aspect of mathematics to learn and teach. Indeed, this is reflected in the high number of persistent fractions errors and misconceptions students make (c.f. Hansen, 2014). One way to start overcoming fraction difficulties is not to only allow students to apply fraction rules without reasoning (Skemp, 1976) but to enable them to deeply interact with fraction representations (Lamon, 2012) and to thus initiate sense-making activities. These sense-making activities support students' conceptual knowledge which in turn is an effective instrument for addressing errors. Supporting students to overcome their difficulties paves the way for them to become proficient in dealing with fractions, a key domain in the area of mathematics education as fractions attainment at elementary level is a predictor for their future mathematics performance (Siegler et al., 2012).

By providing students with exploratory learning activities and thus encouraging reflection and self-explanation, students are supported to abstract information, construct schemata, and hence develop conceptual knowledge (Koedinger et al., 2012). Due to the open-ended nature of microworlds (Healy & Kynigos, 2010) which enable students to manipulate variables, make experiences and discover concepts, students are offered suitable instructional support for exploring underlying principles. Against this background we aimed to investigate whether students' interactions in a learning platform with such a microworld - already shown to improve overall fractions knowledge (Rummel et al., submitted) - helps to reduce common fractions errors and consider how it may do so.

STUDENTS' DIFFICULTIES IN EQUIVALENT FRACTIONS

By observing errors and identifying patterns it is possible to infer common difficulties students have in relation to fraction concepts and use these to address students' misunderstandings and support learning and teaching (Hansen, 2014). We focus here

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 361–368. Szeged, Hungary: PME.

on two significant, overarching difficulties that face students: whole number bias and the shift from additive to multiplicative reasoning.

Although the origins of students' difficulties related to whole number bias are debated (Ni & Zhou, 2005), the natural number reasoning that students draw upon to make sense of rational numbers is oft-cited as the cause of misconceptions (Vamakoussi and Vosniadou, 2010), with these continuing into adulthood (DeWolf & Vosniadou, 2015). With whole number bias, students are likely to interpret the numerator and denominator as separate entities rather than as part of a fraction symbol in its own right that represents a part/whole relationship, measure, operator, quotient or ratio (Kieran, 1993).

Students also often rely on more-familiar additive structures when making sense of fractions (Siemon, Breed & Virgona, 2010) and although students may have learnt their multiplication and division facts, it does not mean that they are necessarily using multiplicative reasoning (Brickwedde, 2011). There are different perspectives on how to support students' transition from additive to multiplicative reasoning, but all agree that multiplicative reasoning supports students' later mathematics work (Brickwedde, 2011; Siemon, Breed & Virgona, 2010). If using additive reasoning, students are likely to apply incorrect 'rules' without considering the invariant proportional relationship between numerator and denominator.

THE ITALK2LEARN PLATFORM

The learning platform was developed during iTalk2Learn, an EU-funded project[1] aimed at supporting students' conceptual and procedural knowledge of fractions. It enables students to learn with exploratory and structured tasks. Students are encouraged to talk to the learning platform, use fraction-specific vocabulary and reflect out loud on their learning. As only the microworld component of the platform was specifically designed to address errors, we introduce it and one of its key functionalities below.

Fractions Lab

Fractions Lab (http://fractionslab.lkl.ac.uk/) is the microworld within the iTalk2Learn platform. Students are encouraged to create and manipulate various fraction representations while carrying out exploratory tasks that challenge errors related to common difficulties such as whole number bias and multiplicative reasoning.

Four types of graphical representations (number lines, area models, sets, liquid measures) are available for students to construct fraction representations. Students are able to manipulate the representations using different tools, for example finding an equivalent by partitioning the representation (Figure 1). When the 'find equivalent' tool is selected, a copy of the original fraction is made and the student can change the number of partitions while the fraction symbol alongside it simultaneously changes.

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[1] The work described here has received funding by the EU in FP7 in the iTalk2Learn project (318051). Thanks to all our iTalk2Learn colleagues for their support and ideas and implementing the learning platform, pre- and post-tests.

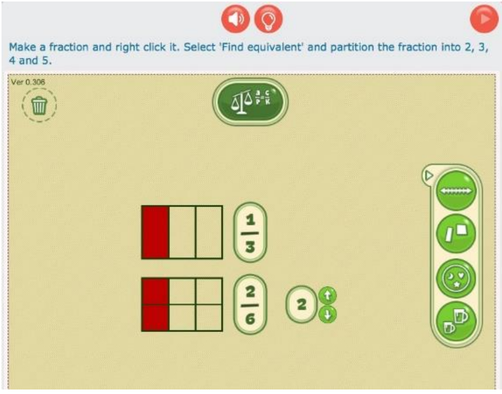

Figure 1: An example task. The student is encouraged to create equivalent fractions by partitioning. Here, two sixths has been made by partitioning one third.

METHOD

In order to investigate how students' errors were addressed after learning with the platform (as compared to before) we conducted a study with 213 students (8-10 years old) in England. Overall, the study took 90 minutes and students had 40 minutes interaction time with the learning platform. Before and after interacting with the platform the students completed a pretest and a posttest including six items measuring their fraction knowledge. We report here upon two items (see Figure 2) addressing errors related to whole number bias and multiplicative reasoning (see Table 1).

Q1. What goes in the box? $\frac{3}{4} = \frac{6}{\square}$	(a) 4	(b) 8	(c) 9	(d) 12
Q2. Which of these is equivalent to 5/6 and has 18 as the denominator?	(a) $\frac{15}{18}$	(b) $\frac{5}{18}$	(c) $\frac{18}{6}$	(d) $\frac{18}{15}$

Figure 2: Two questions related to whole number bias and multiplicative reasoning.

Option	Q1	Q2
(a)	D unchanged	Correct
(b)	Correct	N unchanged
(c)	N + N	N/D confusion & D unchanged
(d)	N x D	N/D confusion

Table 1: Options explained (N = numerator, D = denominator).

In parallel to the pretest and posttest we interviewed a subsample of 12 students in order to get a more detailed insight into their common errors and difficulties and how learning with the platform helped them to overcome these difficulties. The one-to-one interviews were of 30 minutes duration and included questions pertaining to project components including understanding equivalent fractions. As data entry and analysis is ongoing at the time of submission we mainly focus here on the descriptive data in the findings and will present the final results at the conference.

FINDINGS

Although we report only on two questions in this paper due to space constraints, we have evidence on the effectiveness of the overall intervention and in terms of the overall measure of fractions knowledge, the students' knowledge increased (Rummel et al., submitted).

We focus here on two questions (Q1 and Q2) which are related to whole number bias and additive/multiplicative reasoning (the other questions relate to representational errors). There are significant associations between the two errors made in the pretest and posttest (errors from Q1: $\chi 2$ (1, N = 213) = 32. 15, p = .000, errors from Q2: $\chi 2$ (1, N = 213) = 21.96, p = .000). The frequency of the specific error responses exhibited in the questions are shown in Table 2. Indeed, across both questions there was a 6% increase in the number of students providing the *correct* answer and in all bar one response there was between a 2% and 7% reduction in the number of students exhibiting errors.

Option	Q1	Q1 difference	Q2	Q2 difference
(a)	D unchanged Pre 35 (16%) Post 24 (11%)	-11 (-5%)	Correct Pre 112 (53%) Post 125 (59%)	+13 (+6%)
(b)	Correct Pre 132 (62%) Post 146 (68%)	+14 (+6%)	N unchanged Pre 54 (25%) Post 42 (20%)	-12 (-5%)
(c)	N + N Pre 16 (8%) Post 12 (6%)	-4 (-2%)	N/D conf, D unc Pre 29 (14%) Post 15 (7%)	-14 (-7%)
(d)	N x D Pre 15 (7%) Post 7 (3%)	-8 (-4%)	Terminology Pre 10 (5%) Post 11 (5%)	-1 (0%)

Table 2: Frequency of responses for Q1 and Q2 (nil responses not included).

Some errors are common amongst the students in the study. Q1 was answered more effectively with 62% (pre) and 68% (post) of students answering correctly but within the errors exhibited, 16% (pre) and 11% (post) of the students chose an option where the denominator remained unchanged when finding an equivalent fraction. Furthermore, 8% (pre) and 6% (post) added the two numerators given to find a new denominator, and 7% (pre) and 3% (post) multiplied the numerator and denominator of the given fraction. In these three errors we see that the students are treating the numerator and denominator as separate entities and that they do not yet have an appreciation of the invariant relationship between the numerator and denominator. This issue is also evident in Q2 where 25% (pre) and 20% (post) of students did not change the numerator, and 14% (pre) and 7% (post) did not change the denominator. This provides evidence that the students are experiencing difficulties with whole number bias and additive/multiplicative structures.

We also note that in Q2 the students appeared confused with the use of numerator/denominator. Whilst this is rectified in 2c with 7% fewer students selecting the incorrect response, it is not evident in 2d.

We see the biggest reductions in the number of students selecting the 'numerator unchanged' (2b: 5%) or 'denominator unchanged' (1a: 5%, 2c: 7%) options when finding an equivalent fraction. One of the interviewees, Elizabeth, made specific reference to this:

> Elizabeth: [Fractions Lab] helped me, equivalent fractions don't always have to have the same denominator and you can use different [representation] types to make equivalent fractions. I worked out 'find equivalent' eventually. It helped me because [the partitioned rectangle] was kind of like a chart when you split it into small bits. You double it and the denominator becomes bigger but the bits become smaller. It was pretty new to me, we'd only used fraction walls before.

We tentatively suggest that the reduction in students' errors and Elizabeth's comment may reflect a small change in the students' whole number bias and may also reflect a small shift from additive reasoning towards multiplicative reasoning. Elizabeth is reflecting on the role of the 'find equivalent' tool. While representations are partitioned, the fraction symbol is shown contemporaneously. George also referred to partitioning using the liquid measures model.

> George: Because I practised it I got more confident with it and how to find equivalent fractions. I like using the jug because I like measuring. You can split it into parts.

Three students referred specifically to additive or multiplicative structures. Ella refers to Fractions Lab helping her to think about equivalent fractions by seeing the fractions "go up. 8+ 8." Laura refers to both additive and multiplicative structures and Oscar refers just to multiplicative structures.

> Ella: It would go up. 8 + 8.
>
> Laura: You could add 1 and 1 and 1 or 4 and 4 and 4. What you could also do was your times tables.
>
> Oscar: It splits up the rectangle so if you make it into four [partition the rectangle four times] then it is four times four which is sixteen. I had never thought about equivalent fractions like that before.

We tracked these three students' pre and posttests to see if there was any possible link between their comments and attainment on the tests (see Table 3).

Student	Q1 (Pretest / Posttest)	Q2 (Pretest / Posttest)
Ella	Unanswered / Correct	N/D confusion & N unchanged / D unchanged
Laura	Unanswered / Correct	N/D confusion & N unchanged / D unchanged
Oscar	Correct / Correct	N/D confusion & N unchanged / Correct

Table 3: Three students' performance on Q1 and Q2.

For all three students there appears a general improvement across both questions. Ella and Laura answered Q1 incorrectly in the pretest but correctly in the posttest. In all three cases the students exhibited confusion in Q2 related to the numerator and denominator in the pretest but this was not evident in the posttest. However their success in Q1 was not completely mirrored in Q2, with all overcoming their numerator/denominator confusion but two not changing the denominator as required. Q2 is more cognitively challenging than Q1: Q1 requires one operation to be carried out whereas Q2 requires students to know the term 'denominator' as well as carry out the calculation, that contains 'harder' fractions than Q1.

We wonder if the platform requiring the students to use the terms 'numerator' or 'denominator' (and prompting them into use when they were not) may have influenced the students' understanding of the terms. However if this is the case, there is no influence in 2d where there was no change.

DISCUSSION

Students are often fed a narrow diet of representations and interpretations (Charalambous, Delaney, Hsu & Mesa, 2010). However, providing a range of representations such as sets, number lines, area models and liquid measures is paramount to student learning (Lamon, 2012) because these different representations help students understand the underlying fraction concepts (Ainsworth, 1999) and improve conceptual learning. Our findings support the literature on the importance of teaching with a range of representations: Elizabeth explicitly referred to the range of representations, and others to specific representations, as instrumental in supporting them consider equivalent fractions, particularly when the models could be partitioned, e.g. Oscar - rectangle, George - jug.

We believe that the manipulation of the virtual representations is crucial in the students' general thinking-in-change (Hansen, Mavrikis, Holmes & Geraniou, 2015). The affordances that Fractions Lab provides, such as the 'find equivalent' (partitioning) tool also supported some students to think about equivalent fractions in a different way.

Indeed, Oscar and Elizabeth were able to explain how the rectangle became partitioned and the effect partitioning had on the equivalent fraction symbol.

Perhaps more interesting is the relationship partitioning appears to have with the interviewees' additive and multiplicative reasoning. Ella, Laura and Oscar all explained what happened to the numerator or denominator of the symbol as they partitioned. Ella used additive reasoning in her explanation ("8 + 8"), Laura used additive and multiplicative reasoning ("You could add 1 and 1 and 1 or 4 and 4 and 4. What you could also do was your times tables") and Oscar used multiplicative reasoning ("it is four times four"). What we find interesting is that Oscar, who explains the relationship between the partitioned rectangle and the four times table, is more successful in the two test items than those students discussing equivalent fractions in relation to additive structures. We are tentative about claiming anything more from this observation and even that Oscar is using multiplicative *reasoning.* As Brickwedde (2011) reminds us, just because students know their times tables it does not mean they are using multiplicative reasoning. However, we conjecture that the platform (and Fractions Lab in particular) may be a useful resource to support some students' shift from additive to multiplicative reasoning.

Although there is a significant association from pretest to posttest, the errors we have observed seem to be persistent. Yet the improvement observed is worthy of note considering the time the students had undertaking the exploratory tasks. Indeed, we are mindful that the period of interaction the students had with Fractions Lab is very short because the students were additionally undertaking complementary structured tasks.

CONCLUSION

Students' fractions difficulties often stem from whole number bias and the shift from additive to multiplicative reasoning. Despite students working independently on the iTalk2Learn platform for a relatively short period of time, there was a statistically significant overall improvement in their fractions knowledge. Additionally, some students were able to overcome the errors they had exhibited prior to their time on the platform. In light of this, we set out to consider how the platform, and in particular Fractions Lab, might be effectively supporting students to address their difficulties.

We saw around one third of the students making errors related to whole number bias and additive reasoning. We make the case that the 'find equivalent' tool, which enabled partitioning of representations, was instrumental for students being able to notice relationships between equivalent fractions and their numerators/denominators in real time, thus helping them to address their difficulties with whole number bias and the shift from additive to multiplicative reasoning. We are less clear whether the platform's requirement for students to speak aloud and use the terms 'numerator' or 'denominator' contributed to the students' mixed improvement in confusion between the terms.

Further analysis is required to identify the specific tasks that the students undertook. These data will inform our analysis of how the exploratory tasks (rather than the microworld's affordances alone) played a part in addressing the students' errors.

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THE RICHNESS OF POSSILITIES IN ADAPTING A TASK

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Adaptive education aims to meet the needs of individual students. Teachers have to be able to judge if problems lie within the zone of proximal development of each single student and adjust a problem if it does not. This study is a reproduction and extension of the study of Nicol, Bragg and Nejad (2013). In our study, 51 small groups of pre-service teachers were asked to offer suggestions for adapting a problem. The suggestions given were analysed with an extended version of the original coding scheme. Results show that students give more suggestions to make the problem more accessible than to make it more challenging. The background of the pre-service teachers seems to influence the type of suggestions given. In contrast to Nicol, Bragg and Nejad (2013), we found that some of them varied on the big ideas of the problem.

THEORETICAL FRAMEWORK

Dutch primary schools are legally bound to offer adaptive education (van Gerven, 2009), meaning that tasks, instruction and materials are adapted to learners' capabilities and needs. Mooij, Hoogeveen, Driessen, van Hell & Verhoeven (2007) argue by citing Sternberg and Grigorenko (2002) that if education is adapted to the learning level and capabilities of the learner students outperform learners who aren't taught in the way that fits their way of thinking. As Berk and Winsler (1995) write:

> According to Vygotsky, the role of education is to provide experiences that are in the child's zone of proximal development (p. 25)

Getting education to be challenging for each child, standard tasks used in regular education can be adjusted. There are many ways in which the level and complexity of a task can be adjusted and in many ways these adjustments can be clustered and/or classified. In literature several types of classification are used for different reasons, such as evaluating the curriculum or assuring a high quality of assessment. Examples are the iceberg metaphor, learning landscapes, Bloom's taxonomy and the classification used by TIMSS. Each type of classification indicates strategies to adjust tasks.

The purpose of the iceberg metaphor (Webb, Boswinkel, & Dekker, 2008) is to illustrate how mathematical knowledge is constructed on the basis of prior mathematical knowledge, which in general is less formal and more concrete at primary school level. The central concept of the iceberg metaphor is that in order to be able to work with formal concepts like the fraction ¾ in a correct and flexible way, a large amount of 'under water knowledge' is needed, the so-called floating capacity. In the case of ¾ this is knowledge like dividing 3 pizza's over 4 persons. De Corte (1995) mentions that learners differ by the time they need to build a base of informal experiences. More capable learners can pass much quicker to higher levels of formality. Consequently soon after introducing a new topic, learners already differ in the level of formality of the assignments which are in their zone of proximal development. Tomlinson (2001, 2003) also asserts tasks may be defined on a continuum from concrete to abstract and a given problem is made more concrete by giving students manipulatives. It can be concluded that the level of formalness, for example by giving manipulatives, influences the level and complexity of a task. Another way of organizing learning content are the so-called ‘learning landscapes’ (Fosnot & Dolk, 2001). A learning landscape contains models, strategies and ‘big ideas’. In a learning landscape, learners move upward along these constituents. It is not necessary to understand each single constituent, since different paths are possible, but there should be a path from the bottom upwards; this idea compares to the idea of the floating capacity of the iceberg. Learning landscapes make clear that big ideas are essential and therefore one should keep in mind which big idea is addressed while adjusting a task.

Bloom’s taxonomy (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956) is hierarchical with six steps: knowledge, comprehension, application, analysis, synthesis and evaluation. The hierarchy of the taxonomy has been criticised, especially that of the higher-order thinking skills (the last three) which can be conceptualised as distinct but parallel (Long, Dunne & de Kock, 2014). Revisions of the taxonomy are widespread. Krathwohl (2002) for example gives the following categories: remember, understand, apply, analyse, evaluate and create. From these taxonomies it can be concluded that tasks can be adjusted by asking questions of different categories of the taxonomy. A question to evaluate (why is it that ….? or Is it a coincidence that … ?) or to create (think up of a similar problem for other children) are examples of higher order tasks and therefore more suitable for strong performers.

TIMSS and CAPS classify questions and their answers into the following categories: knowledge, routine procedures, complex procedures and problem solving (Long, Dunne & de Kock, 2014). One aspect of the difference between routine and complex procedures is the numbers used in a task, since students’ familiarity of numbers affects their performance (Blessing & Ross, 1996; Ebersbach, M., Luwel, K., & Verschaffel, L., 2015). Another aspect is the number of facets in a problem, these include the number of steps required to solve it, the number of variables, or the number of different skills to be employed (Little, Hauser, & Corbishley, 2009). Little et al. (2009) state that indirectly the number of facets can be decreased by providing extra scaffolding. Blessing and Ross (1996) found that the amount of correlations between a problem’s content and its deep structure influences the level and complexity of a problem. It can be concluded that the familiarity of numbers, the number of facets in a problem, whether or not scaffolding is provided and the correlation between content and deep structure, influences the level and complexity of a task.

Nicol, Bragg & Nejad (2013) argue that learning to design and adapt problems for mathematics teaching that meet the diverse needs of students and maintaining the richness while teaching are not a trivial endeavour both for pre-service as for experienced teachers. In their paper, they examine how preservice elementary teachers adapt a task. The original coding scheme of their paper is given in Figure 1 (more accessible) and Figure 2 (more challenging) in the first two columns. Nicol, Bragg and Nejad (2013*)* identified these categories on the basis of the work of their participants. In the third column we added the conclusions drawn from the different types of classifications which are comparable.

a vary mathematical content	introduce numbers that are more familiar	TIMSS, familiarity of numbers
b vary context	Use fewer words and more diagrams	Iceberg, role of context
	Use manipulatives	
c vary question asked	Work forward	Iceberg, role of content
	Decrease number of problem steps	TIMSS, number of facets
	Provide structured support	TIMSS, scaffolding

Figure 1. Alternatives for mjaking the problem more accessible (original version)

a vary mathematical content	introduce different fractions	TIMSS, familiarity of numbers
b vary context	include more or extraneous information	Iceberg, role of context
c vary question asked	provide open-ended questions	Bloom’s taxonomy
	construct original questions	Bloom’s taxonomy

Figure 2. Alternatives for making the problem more challenging (original version)

In these figures some of the adaptions we mentioned before, are missing: the number of facets is only present in making the problem more accessible, in both schemes varying on the big idea is missing as is the level of formalness and/or concreteness, the number of facets is missing in making the problem more challenging.

METHOD

Participants

In this study, 51 groups of a maximum 4 students participated. All participants were pre-service elementary teachers. There were two types of students. The first group of students included 114 regular students who started their teacher education soon after they had finished secondary education. The second group of students consisted of 20 part-time students who study during weekends and in the evenings. On average these students were older than the regular students and in many cases they had a family. All part-time students had a bachelor degree, but not in education.

Procedure

All participants were students from a primary teacher education institute in a small city in the Netherlands. They were taking an obligatory course about adjusting education to the needs and capabilities of individual learners. The study was performed during the first meeting of the course, before any information whatsoever (besides the information in the curriculum overview) was given about the course. Participants were told that they were allowed to work together in pairs (or in case of an odd number of participants in a group of 3). Because some pairs could hear one another's discussion those pairs worked together. Participants were given a problem about fractions. They were asked to give suggestions how to make the problem both more accessible and more challenging. Participants were given as much time as they needed.

Task

In this study the task used by Nicol, Bragg and Nejad (2013) was translated into Dutch.

> *Three Hungry Monster Problem:* Three tired and hungry monsters went to sleep with a bag of cookies. One monster woke up, ate 1/3 of the cookies, then went back to sleep. Later, the second monster woke up and ate 1/3 of the remaining cookies then went back to sleep. Finally, the third monster woke up and ate 1/3 of the remaining cookies. When she finished there were 8 cookies left. How many cookies were in the bag originally?

Data analysis

The analysis scheme used by Nicol, Bragg and Nejad (2013) formed the basis of the analysis, but the scheme was extended if necessary. The number of forms that contained a suggestion in each option in each category were identified and we examined whether the type of student influences the type of given suggestions.

RESULTS

In total 51 forms were collected, 43 from regular students and 8 from part-time students. Only 39 of the forms of the regular students contained suggestions for making the problem both more accessible as more challenging. Two forms didn't contain suggestions given for making the problem more accessible and one form didn't contain suggestions given for making it more challenging.

We extended both schemes (Table 1 and 2) by adding a category d 'vary on theme of big idea'. Suggestions were made about going to another subdomain like percentages or ask for the limit of repeating the process of awakening, so we added a category e 'vary subdomain' (see Table 2). Besides extending the scheme for challenging alternatives by adding more categories, we also added possibilities within the existing categories. For example, the question 'how many more times do the monsters have to wake up, before there's 1 cookie left for each monster?', was put in the new option 'make the problem a more puzzling task' within the category a. Besides this addition we added the possibilities of starting with more cookies, of working with different fractions within the task, and of having a result which is no longer an integer within category a. Within category b we've added the possibility of making the task more formal. In the original scheme nothing was said about raising instead of decreasing the number of problem steps, which was in category c of the 'more accessible version'. Because in the 'more challenging version' category c contains different kind of adaptions, we see it as a form of including more information, category b (for the complexity is raised by raising the amount of information, not by raising the complexity of the question itself).

Suggestions to make the problem more accessible

In Table 1 results are given for the task of making the problem more accessible. One form contained the suggestion to first have learners practice the multiplication table of 3 before trying to solve the problem. In contrast to the regular students, significantly less part-time students varied the task by changing the numbers into more familiar numbers ($t(13{,}07) = 3{,}40$; $p = 0{,}005$) or suggested to decrease the number of problem steps ($t(39{,}00) = 4{,}58$; $p < 0{,}001$). Part-time students more often suggested to change the order of the problem ($t(12{,}39) = -3{,}97$; $p = 0{,}002$) and they gave more different options (maximum was 3, see Table 1) within the category of varying the question asked ($t\ (46) = -2{,}26$; $p = 0{,}028$).

			% of the forms on which the suggestion is made	
	Category	Options within category	Regular	Part-time
a	vary mathematical content	introduce numbers that are more familiar	62,5	12,5
b	vary context	Use fewer words and more diagrams	30,0	25,0
		Use manipulatives	12,5	37,5
c	vary question asked	Work forward	30,0	87,5
		Decrease number of problem steps	35,0	0,0
		Provide structured support	10,0	50,0
d	vary theme of big idea	Have the monsters eat 1/3 of the total and not 1/3 of the remaining cookies	7,5	37,5

Table 1. Results of the alternatives for making the problem more accessible

Suggestions to make the problem more challenging

In Table 2 the results are given for the task of making the problem more challenging. Some forms contained the suggestion to have the parents share the remaining cookies equally. It is an interesting suggestion, because now the fraction is taken two times from the same amount, in contrast to taking the fraction of the remaining, which was done by the 'little' monsters. We choose to put this suggestion in category b instead of d, because no explicit remark was made about this difference between parents and children and it therefore just might be another form of adding facets.

		% of the forms on which the suggestion is made	
		Regular	Part-time
a vary mathematical content	introduce different fractions or start with more cookies, work with different fractions within task	81,0	37,5
	have a result which is no longer an integer	4,8	0,0
	make task more like a puzzle	2,4	0,0
b vary context	include more or extraneous information	40,5	0,0
	make task more formal (formulated only in mathematical symbols)	9,5	50,0
c vary question asked	provide open-ended questions	2,4	0,0
	construct original questions	2,4	0,0
d vary on big idea		11,9	12,5
e vary subdomain	ratio (like percentage, decimal numbers)	4,8	0,0
f vary goal	make the learner visualize the problem for others	0,0	12,5

Table 2. Results of the alternatives for making the problem more challenging

What kind of suggestions did we categorize as variations from the big idea? One example are questions concerning the amount of cookies the second and third monsters are still entitled to. Another example is having each monster eats the same amount of cookies but now ask which part this is of the amount of cookies that are there when the monster awakens (and have children see that this is a different fraction each time; 1/3, 1/2, 1/1). Another variance found was the question how many times you have to take 1/3 of the remaining before you are below ½ of the total. One form had an added question whether or not it is a coincidence that $1/3 \cdot 1/3 \cdot 1/3 = 1/27$. An interesting detail is that one form contained 4 different kinds of variations concerning the big idea.

There were also some suggestions that made us wonder why they were considered more challenging by the students. Examples of suggestion we ourselves would have put in making accessible are (1) a question asking whether or not the cookies are divided fairly, (2) have the second and third monster share equally instead of taking 1/3 of the remaining cookies (and this contradiction is mentioned explicitly).

As it was with the more accessible suggestions, part-time students suggest significantly less often to vary the task by varying the mathematical content ($t(8,47) = 2,39$; $p = 0,042$). Having this result in mind it is not surprising that part-time students also suggested significantly less often to make the problem more challenging by adding redundant information ($t(41,00) = 5,28$; $p < 0,001$).

CONCLUSION AND DISCUSSION

In this study the number of participants was much larger than in the original study of Nicol, Bragg and Nejad (2013). We found suggestions which were similar to those of the original study, but we also found new suggestions amongst others about variations from the big idea. This led to extended coding schemes. In this study there were two different types of students. It turned out that they differ in the suggestions they give.

Both groups of students had relatively more different categories of suggestions for making the problem more accessible than for making it more challenging. A reason might be that quite a few participants found the original problem quite challenging. It would be interesting to investigate whether a different result would be obtained if preservice teachers had to adapt a problem which they find easy to solve. The result that different types of students give different kind of suggestions, raises the question whether more inexperienced teachers, inexperienced both in teaching and in knowledge of life, focus more on superficial aspects like changing the numbers or the amount of information, while more experienced teachers focus more on deeper aspects like the order of the problem (forward or backward). It is desirable to extend the research to in-service teachers.

In this study a suggested adaption was to ask how long it will last before all cookies are completely eaten if each time 1/3 of the remaining is taken away. It is interesting to see that this in fact is a mathematically interesting question, but it is a misplaced question in this context. At the moment a monster awakes and finds just one cookie left, it will eat all of it instead of eating just 1/3. It is a nice example of vertical mathematizing (Freudenthal, 1991): for strong learners, the context can be taken away and the problem can be seen from a purely mathematical point of view, leading to new mathematical interesting questions.

We didn’t judge the different suggestions given, but in a follow up study this might be done. For example: though adapting the task by making the numbers more or less familiar affects succeeding the task, it is discussible whether or not it makes the task more challenging in terms of insight. Miscalculations are more likely to occur and working memory is much more loaded, but if one understands how to solve the problem with more familiar fractions, there’s no new insight needed. Another question is whether making the task less concrete raises or lowers the level of complexity. Piel and Suchart (2014) found in their study of social class related differences that the choice of the context influences the level of intelligibility. Realistic items had more differences due to their cognitive ability, class and sex than pure items.

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AN ANALYSIS OF HOW TEACHERS' DIFFERENT BACKGROUNDS AFFECT THEIR *PERSONAL RELATIONSHIPS* WITH CALCULUS CONTENT IN ENGINEERING COURSES

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This paper continues work initiated in Hernandes-Gomes & González-Martín (submitted), which examines whether engineering teachers' views of Calculus are influenced by their different academic and professional backgrounds, and how these views impact their teaching practices. We follow an institutional perspective, using Chevallard's Anthropological Theory of the Didactic (ATD), in particular the notion of personal relationship. Our data suggest that different academic and professional backgrounds may influence engineering teachers' personal relationships with the content of Calculus courses, thus resulting in different practices.

INTRODUCTION

Calculus is a first-year course in most science, technology, engineering and mathematics (STEM) university programs. Calculus courses are meant to introduce the tools students require in their more advanced courses. However, in the case of engineering programs, Harris, Black, Hernandez-Martinez, Pepin, Williams, and TransMaths (2015, p.321) think that "whichever way the mathematics within engineering courses is taught, there are problems for some students". The literature seems to indicate that this issue is generalised to other programs (Rasmussen, Marrongelle, & Borba, 2014), and could contribute to a student's decision whether to continue pursuing a STEM degree or not (Ellis, Kelton, & Rasmussen, 2014). Not only do teachers need to address students' difficulties in learning Calculus notions, they also must acknowledge "the need for bridging the gap between mathematics and sciences" (Dominguez & de la Garza Becerra, 2015). When it comes to teaching mathematics in professional programs, Christensen (2008, p.131) has pointed out that "it can be quite difficult to connect the abstract formalism of mathematics with the necessary applicable skills in a given profession", and that this could produce a "gap in the students' ability to use mathematics in their engineering practices".

Research at the undergraduate level has identified many of the difficulties students face in their first-year mathematics courses at university (Rasmussen *et al.*, 2014). However, more research is needed to ascertain how these difficulties are taken into account through teaching practices at the tertiary level (Rasmussen *et al.*, 2014). Indeed, the literature has identified variability in these practices. For instance, Wagner and Keene (2014) analysed the practices of two university mathematics teachers on the first day of an undergraduate differential equations course. Each instructor held a doctorate in mathematics and possessed more than fifteen years of experience teaching at the university level. Their results showed that both "mathematics professors

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 377–384. Szeged, Hungary: PME.

addressed the first day's mathematical content in different ways, and […] that how the professors interpreted the curriculum is an important and appropriate way to consider how the enacted curriculum may have affected student learning" (p.330). We also cite Pinto (2013), who, in analysing two university teachers' implementation of the same lesson plan in a course on infinitesimal Calculus, showed "how different beliefs and attitudes, different goals and reliance on different resources resulted in two substantially different lessons." (p. 2424) These two works seem to indicate that university teachers may rely heavily on their own experience and their own vision of mathematics and teaching. For this reason (among others), further research focusing on university teachers' knowledge and beliefs (particularly when it comes to Calculus) is necessary (Eichler & Erens, 2014). We intend to address this lacuna using an institutional perspective. In the context of engineering courses, Calculus can be taught by teachers with very different training and professional experience; we believe that an institutional perspective will shed light on the origin of these teachers' Calculus instruction practices.

Our work also contributes to an emerging field of research on the teaching and learning of mathematics in engineering. Artigue, Batanero and Kent (2007, p.1031) previously pointed out that "there have been very few studies of the different ways that mathematics and engineering students think about mathematics." However, it is also necessary to study how instructors with a background in mathematics think about mathematics and their teaching, as opposed to instructors with a background in engineering. In Hernandes Gomes & González-Martín, (2015) we addressed this issue by identifying differences in the way two teachers with different academic backgrounds approach mathematical topics in engineering programs. These results led us to examine this topic more closely, drawing on a larger pool of teachers with different backgrounds and following an institutional perspective (Hernandes-Gomes & González-Martín, submitted).

THEORETICAL FRAMEWORK

As stated above, we are interested in studying how teachers' visions of mathematics and their teaching vary according to the instructors' different training and professional experiences, particularly with respect to the teaching of Calculus in engineering programs. Because we wish to use an institutional approach, we employ Chevallard's (1999) Anthropological theory of the didactic (ATD).

ATD sees mathematical (communal) knowledge (*savoir* in French) as the product of institutional human action; it is something that is produced, used, taught, or more generally, transposed within institutions (Bosch & Chevallard, 1999). An institution is defined as a social organisation *I* that allows, and also imposes on its subjects, ways of doing and thinking proper to *I*. One key notion of ATD is that of *praxeology*, which allows for the modelling of social practices in general and mathematical activity in particular. ATD also proposes that "any institutional practice can be analysed, from different points of view and in different ways, through a system of tasks relatively well circumscribed." (p. 84) Every task can be tackled or accomplished using one or more techniques; ATD postulates that institutional activity is composed of a wide range of tasks that are carried out according to institutionalised "ways of doing". In this sense, the institutional relationship with an object *o*, for a given position *p* within the institution *I* ($R_I(p, o)$), is defined through the set of tasks accomplished by individuals occupying this position, using given techniques. As a consequence, by carrying out given tasks in various institutions to which an individual belongs simultaneously (or has belonged successively), his or her *personal relationship* with a given object emerges and is constantly remodelled. Techniques are explained by given discourses, called *technologies*, which belong to branches of knowledge called *theories*. This set of task/technique/technology/theory forms a *praxeology*. The *personal relationship* includes elements such as 'knowledge', 'know-how', 'conceptions', 'competencies', 'mastery', and 'mental images' (Chevallard, 1989, p.227).

To illustrate the application of these notions in the context of our research, let us consider a faculty of engineering (*E*) comprising several positions, including teacher (in various departments) and student. Figure 1 illustrates the following scenario. An individual can occupy the position of student in a faculty of engineering (*E*), learning limits in the context of praxeologies that exist under the restrictions of $R_E(s, \lambda)$. This notion is approached in a particular way in professional courses, due to the existence of specific praxeologies ($R_E(s, \Lambda)$), which may modify the individual's *personal relationship* with said notion. The same individual can later occupy the position of engineer at a firm (*F*). It is likely the praxeologies present in this new environment ($R_F(e, \Lambda)$) will further remodel the individual's *personal relationship* with Calculus notions. A second individual can occupy the position of student in a faculty of mathematics (*M*), developing a different *personal relationship* with limits, first in introductory Calculus courses (under the restrictions of $R_M(s, l)$) and later in more advanced courses ($R_M(s, L)$). These two individuals then go on to teach Calculus in a faculty of engineering (*E*), a new position for both of them. Even though they now occupy the same position in the same institution (under the restrictions of $R_E(t, \lambda)$), their *personal relationships* with limits are likely very different, which could have an impact on their teaching practices.

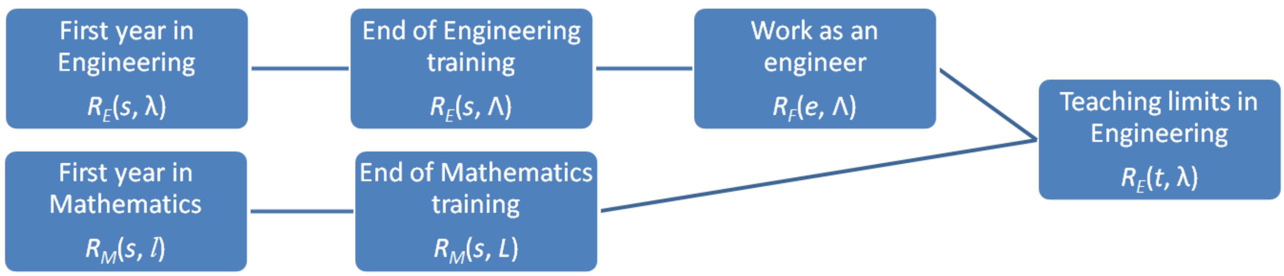

Figure 1: Different paths to becoming a Calculus teacher in engineering

We believe that ATD offers an interesting lens through which to observe and analyse these phenomena, allowing us to identify variations between different teachers' *personal relationships* with the notions they teach. This could help explain teachers' divergent practices and the various choices they make in preparing their courses.

METHODOLOGY

In September 2015, we interviewed six university teachers with different academic backgrounds (Figure 2). They all had been teaching first-year Calculus in university engineering programs in São Paulo, Brazil, for at least 14 years. One month before the interviews, each teacher received a questionnaire on their academic and professional backgrounds, which allowed us to categorize their profiles. For this paper, we compare the main results from our interviews with teachers T4 and T5.

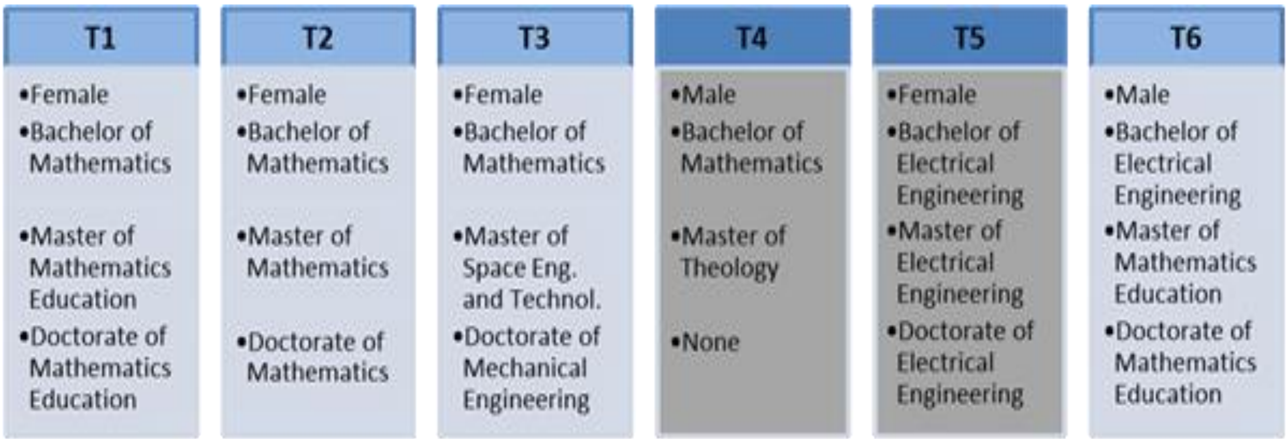

Figure 2: Profile of six university teachers

Both instructors teach Calculus in first-year engineering courses at the same university. T4 has taught Calculus I and II for 45 years; his professional experience is limited to university teaching. On the other hand, T5 has been a Calculus instructor for 14 years and has taught Calculus I at her current university since 2011. She also worked as an electrical engineer for 36 years, both in-house and as a consultant. At their university, Calculus I covers functions, limits and derivatives, ending with rate of change and optimisation problems. The course is organised around the classical praxeology of introduction of definition, properties, theorems, exercises and some applications.

All the interviews were audio recorded and transcribed. They took place at the teachers' workplace in a room with only the interviewer (first author of this paper) and interviewee present, on a day chosen by the interviewee. The questions were designed to establish the main aspects of the teachers' vision of Calculus and its teaching, reveal how their academic and professional backgrounds influence their practices (including specific elements that might influence their *personal relationship*) and identify the choices they make in preparing the course and student exercises, taking into account the fact this Calculus course is geared towards engineering students. After the transcription was complete, the teachers' answers were coded, which allowed us to categorize the data to develop our analysis. Figure 3 lists the elements we discuss in this paper.

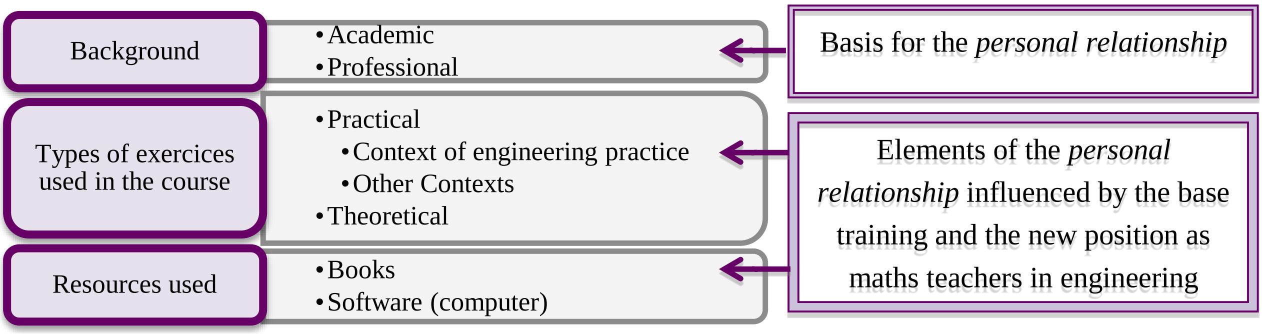

Figure 3: Final categories and subcategories

DATA ANALYSIS

Both teachers were asked how their academic and professional backgrounds influence their course preparation. They were also asked about their choice of student exercises and about the books and resources they select to prepare their course and use in class. Their responses were as follows:

> T5: In the contextualisation, the application of exercises, then it influences quite a bit. […] And in electrical engineering, so to speak, you model […] circuits, and components, through mathematics. […] Let's say I have […] an integrator circuit, I have to know what [it] does, I need to know what is the integral, how it is that I throw a pulse and it starts the integral […].With an equation, a second order filter becomes a polynomial equation of second degree. I cannot dissociate one thing from the other. [...] Because that is how I see Calculus for engineering: it's modelling. I look at an oscilloscope and I see a function, and in the same way that I see a function, I think of an electric signal associated with that function. Because it's my way of understanding engineering and mathematics. Mathematics helps me with physics and Calculus problems. That's why I went into teaching Calculus.

On the other hand, in answering these and other questions, T4 revealed that the difficulties he had encountered with mathematics as a graduate student continue to influence his choices as a teacher:

> T4: At the beginning I had a lot of difficuly, because I almost always had to teach something that I hadn't mastered myself. So I had to study a lot. I got different books; I've always been a bit of an autodidact. […] But I could learn new things on my own, [I] overcame my own difficulties and I pass [this new knowledge] on to [my students]. […] So, books, they are essential. I always say [to my students]: "You cannot be an engineer if you don't have […] a library [at home]. […] You need to have books on the basics as well as specific [professional] books. Because at times you'll have a look at your notes, and you won't find [what you're looking for], and then you'll go to your book." […] I value books. Very much.

We see in these excerpts that both have a very different *personal relationship* with the content of the Calculus course they teach. T5 can clearly relate the content of the course to her experience as an engineer; she has been exposed to praxeologies that allow her to connect this content with the practice of an engineer. On the other hand, T4, while he holds a Bachelor of Mathematics, seems to have a *personal relationship* with

Calculus that is closer to that of a student than a mathematician. This could be because he started teaching immediately after he received his degree, and is therefore lacking experience as a mathematician and engineer.

The teachers' academic and professional backgrounds also seem to influence the amount of time they dedicate to exercises that are specifically relevant to engineering. T4 stated he finds it challenging to select contextualised exercises, due to the fact he is not an engineer and because first-year students are not familiar with advanced engineering concepts. To overcome this limitation, he usually consults his colleagues who teach professional courses on ways to use Calculus course content to maintain students' attention: "This thing here, there's [an application] in engineering, in the professional courses, where you will use this calculation. Even myself, I don't know it very well, but this calculation will be used there, so I believe it's better for you to learn it now, because if not, you'll have difficulties there." We see that he has not participated in any praxeology in which he used the notions he now teaches, and although he seeks advice from his colleagues, he seems to have a superficial understanding of how to apply Calculus directly to engineering practices. At the other end of the spectrum, T5 said that she always explores how exercises apply to engineering: "I have a look at the exercise and I see where it can be applied. I already give a contextualisation for the exercise." Once again, her background in engineering, as well as her professional experience in the field, seem to have given her and understanding of how the notions she teaches may be applied to engineering tasks. This topic also arose at another point during the interview:

> T5: I tell [students] that the exercises I give in class have to be harder than the ones on the exam. Because they can do exercises. […] But like any teacher, you have a preference for a certain type, type of function and, of course, I'll direct things towards whatever is more applicable to electrical engineering.

In contrast, T4's approach to exercises for engineers seemed more closely related to praxeologies of mathematics courses:

> T4: Always more practical, since it's [a course in] engineering. For instance, yesterday I was trying to justify the first fundamental limit $\left[\lim_{x\to 0} \sin x/x\right]$, and I said: "Now I'll make a justification for engineers." […] Then I made a table […] and a student asked me: "[…] there isn't an algebraic proof, professor?" Yes, there is. Now I'm going to do a proof that combines many things, it has geometry, algebra, and trigonometry. I would say that this proof is more for mathematicians, but not very… it's mostly for an average mathematician. [...] But it's more practical, I believe that in engineering, theory must be minimized, as much as possible.

Again, we see that T4 seems to engage in praxeologies involving only basic mathematical notions from the Calculus course, even though he believes they are "more practical". When asked about the importance they ascribe to theorems and proofs, the teachers responded as follows:

T4: Depending on the theoretical level of the proof, I believe that it could be useful, because it enriches reasoning capacities. For example, if the engineering student is interested in pursuing his studies after graduating, like [in] a master's degree, a doctorate. If he wants to be an engineer that makes projects […], he needs to know the theory well. […] Now depending on the level [of his professional activity], he may not use it. But I think the right thing is to do it even if it they won't use it.

T5: I usually do the following, when the proof is pure mathematical manipulation, where you have to manipulate the equations to get the answer you want, I just project it in the data projector and bye. When the proof follows a way of thinking […] I do it, even if it entails a lot of mathematical manipulation, but the concept is in the proof. But [when the proof] is just about doing calculations [I don't do it].

Once more, we see a meaningful difference between T4's and T5's *personal relationship* with the content of their Calculus course. T4's relationship seems influenced by his background in mathematics, which is perhaps why he considers proofs to be important for engineering students even though they may not use them. However, T5's position appears to be shaped by her background in engineering; she identifies those proofs that may help students better understand the mathematical notions they will need in their future practice and distinguishes them from proofs that only offer manipulations and do not enrich an engineer's profile.

FINAL CONSIDERATIONS

Our data, in accordance with those presented in Hernandes-Gomes & González-Martín (submitted), suggest that different academic and professional backgrounds could influence Calculus teachers' practices in engineering programs. When different teachers have belonged to different institutions and participated in different praxeologies, they seem to use different approaches when connecting Calculus notions with engineering practices and deciding which content or skills are most useful for a future engineer. We see that T4 and T5 each possesses a different set of 'knowledge', 'know-how' and 'competencies', among others. This results in different *personal relationships* with the notions taught in their Calculus courses and, thus, with their practices. The gap identified by Christensen (2008) between mathematical notions taught in class and the practical skills required by engineers seems to be addressed more fully by T5 than by T4.

As noted in the introduction, there is need for further research on how engineering students think about (and use) mathematics (Artigue *et al.*, 2007). However, this could be strongly influenced by how their mathematics courses are taught. In this sense, it is also necessary to develop studies that investigate how teachers with different backgrounds plan and organise their engineering courses. Our work intends to contribute to this strand of research. We plan to analyse the results derived from the interviews with T1 and T2 (neither of whom have a background in engineering) and develop a global analysis and comparison of our six participants. This will be the subject of future publications.

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CREATIVITY WITHIN SHIFTS OF KNOWLEDGE IN THE MATHEMATICS CLASSROOM

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In this study creativity is taken in its most common meaning: the ability to transcend traditional ideas, rules, patterns, relationships, and the like, and to create meaningful new ideas. During the last few years, we have been investigating knowledge shifts among different settings in inquiry-based classrooms. The goal of the current study is to investigate creative mathematical thinking within the shifts of knowledge in an inquiry-based mathematics classroom. We demonstrate the creativity within shifts of knowledge via a whole class discussion and a group work. The shifts of creative knowledge started with an individual student and often served as a kind of milestone in the inquiry of the topic.

BACKGROUND

Our research is coordinating Abstraction in Context (AiC) and Documenting Collective Activity (DCA) in order to empirically investigate progress of mathematical knowledge among different settings in the same classroom, as they unfold in the same lesson, resulted in the emergence of several new theoretical constructs (Authors, Year). A *knowledge agent* is a member in the classroom community who initiates a new idea, which subsequently is appropriated by one or several other members of the classroom community. Thus, when a student in the classroom is the first one to express a new idea according to the researchers' observations, and later others in classroom express or use this idea, then the first student is considered to be a knowledge agent. We call these other students *followers*. To empirically identify a student as a knowledge agent, the researchers need to point at her or his follower/s. A student can follow an idea by repeating it, elaborate it, or object to it. The follower's action may occur immediately after the one of the knowledge agent but it may also take place later during the same discussion, or during a later discussion and/or in a different social setting in the class.

The term *knowledge shift* relates to the spread of ideas in the mathematics classroom. A first shift takes place between the knowledge agent and the follower/s. However as other students take part in the discussion, more shifts may become evident. Note that we do not claim that "the same" piece of knowledge is duplicated in the minds of students in the discussion. Rather, each individual appropriates the ideas that are brought up in the collective. An idea can be shifted from a group to the whole class (uploading), within the whole class, within a group, from a group to another group, or from the whole class to a group (downloading).

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Creative mathematical thinking

There is increasing interest among researchers and policy makers concerning the role of creativity in mathematics education, and how it may be fostered (e.g., Singer, Ellerton, & Cai, Eds., 2013). When referring to creativity in school, it is usually considered as a relative phenomenon. In other words, students' ideas will be considered creative on the basis of their contribution to the mathematical knowledge of the class or the group. As Leikin and Pitta-Pantazi wrote: "Creative ideas are those that are considered by the reference social group as new and meaningful in a particular field" (2013, p. 161).

A few studies were conducted where whole class discussion served as data source in an attempt to evaluate the collective creativity of the group (e.g., Levenson, 2011). The current paper takes also a socio-cultural approach to analyse classroom episodes. In addition, we consider the distinction made by Lithner (2008) between *imitative reasoning* and *creative mathematical reasoning*. Imitative reasoning can manifest itself as either memorized or algorithmic. Lithner claims that creative mathematical reasoning satisfies three criteria:

> 1. *Novelty*. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created. 2. *Plausibility*. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible. 3. *Mathematical foundation*. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning (Lithner, 2008, p. 266).

We link the notions of knowledge agent, follower and shift of knowledge in the mathematics classroom to students' creative reasoning. Specifically, we ask: What are the mutual relations between knowledge agents and their followers in the classroom and the creative ideas within mathematical knowledge shifts in the classroom?

METHODOLOGY

A 10-lesson learning unit in elementary probability was designed, implemented, observed and video-recorded in several eighth grade classes. The current study is based on data collected during Lesson 4 of the unit in one class. In this class, a camera was focused on whoever spoke during whole class discussions and on a specific group of three girls during group work. The learning unit consists of a sequence of activities that was purposefully designed to offer opportunities for constructing knowledge and establish practices.

Two problems were at the heart of Lesson 4 (see Figures 1 & 2).

A Hanukkah dreidel (a four-sided top with one of the letters N, G, H, and P on each side) was spun 100 times. Mark approximately, on the chance bar, the chances of the following events:

A. The dreidel will fall 100 times on the letter N.

B. The dreidel will never fall on the letter N.

C. The dreidel will fall on N between 80–90 times.

D. The dreidel will fall on N between 20–30 times.

Figure 1: The Dreidel Problem for the whole class discussion

A coin was flipped 1000 times. Mark approximately, on the chance bar, the chances of the following events:

A. The coin will land 1000 times with heads facing up.

B. The coin will land with heads facing up between 450-550 times.

C. The coin will land with heads facing up between 850- 950 times.

D. The coin will fall 1000 times with tails facing up.

Figure 2: The Coin Problem for groups' work

The first step in our analysis consisted of an a-priori analysis of the two problems, to identify relevant knowledge elements for working on the two problems. The students were expected to use the following knowledge elements constructed during their work on previous tasks:

Eu Uncertainty is inherent in probability problems.

Ee There are expected probability values for each event. Hence, probability is amenable to mathematical reasoning.

Em Results of multiple experiments accumulate to the expected probability value of an event.

The following knowledge elements were expected to be constructed by the students during the work on the two problems:

Ere The probability of a simple event is different from the probability of a composite event that consists of a repetition of the simple event.

Ed The probability of a composite event that consists of repetitions of the same simple event decreases with each repetition.

Era If a given range of values includes the expected value, then the probability of falling into this range is high.

Next we took the transcript of the lesson and parsed it to episodes. All episodes were further analyzed, and we choose a small number among them for presentation. In each episode we identified students who raised new ideas, and marked these students as *potential knowledge agents*. Later, we reanalyzed the transcript to identify whether elaborations, continuations, or objections to these ideas were raised by other students - evidence that a different student followed the idea raised by the potential knowledge agent. If indeed such a student was identified, we consider her as a follower. Hence the first student becomes a knowledge agent, and we conclude that the first shift of this idea (new knowledge) took place. We then analyzed the knowledge that was shifted through the creativity lens – we examined the students' contributions to see if there is evidence for the creative reasoning criteria, and hence this contribution can be identified as creative.

FINDINGS

In the lesson the concept of *chance bar* (a segment between 0 and 1 on which one can mark the probability of an event) appeared. In the Dreidel Problem and the Coin Problem (Figures 1 and 2), the chance bar has mainly a qualitative meaning. The teacher opened the lesson with a whole class discussion in which she asked her students for examples to remind them of the meaning of a few relevant concepts: the chance bar, the meaning of this segment's edges: 0 for the probability of an impossible event, 1 for the probability of a certain event. She asked a few students to mark the probability of various events on a chance bar on the blackboard. Then they started working on event A of the Dreidel Problem (Figure 1). The teacher asked Itamar to mark the probability of event A on the chance-bar. Itamar made a mark for A close to ¼, but then expressed a dilemma: "It is supposed to be impossible".

Argumentation in class – Episode 1, event A - Guy's idea of decrease

68	Teacher	What do you have to respond to that? (Repeats Itamar's argument that there are 4 letters but N is just one letter) Guy, how would you answer him?
69	Guy	There is, like, each time that you spin there is, like, 4 letters it can fall on, so each time it divides again by 4, the chance (Teacher: yes), and the chance decreases, it decreases each time that you spin that it will fall again on the same letter.
70	Teacher	So you are reinforcing him?! You are saying that the mark is correct?!
71	Guy	No, you have to lower it.
72	Teacher	Because…?
73	Guy	Because each time the probability is much smaller, when you spin twice and it falls on the same letter – the probability decreases.

In this episode Guy raised a new idea concerning the probability of event A as repeating event (69, 71 & 73). Guy's idea is mathematically correct. In fact, Guy (69) expressed the knowledge elements **Ere** (The probability of a simple event is different from the probability of a repetition's composite event) and **Ed** (The probability of a composite event which consists of repetitions of the same simple event decreases with each repetition). These are innovative ideas in the framework of this classroom, and we consider them to be an example of creative reasoning. In this episode we still do not have evidence if Guy is a knowledge agent, because we do not (yet) have followers.

After this episode, the class continued to discussed event A, and also events B and C. Because of space constrains we will omit these and proceed with the class to event D.

Argumentation in class – Episode 2, event D

128	Teacher	Let's look at event D. D says, a top is spun 100 times, it will fall on N between 20 to 30 times. What do you think? We will spin the top 100 times; how many times will it fall on N, between 20 and 30 times. [To Eliana] Come, you haven't marked yet. [Eliana approaches the board and marks D close to the middle of the chance bar].
129	Teacher	Adin, what is your opinion, what do you say?
130	Adin	I think that it is approximately 30%.
131	Teacher	That means that you agree with what Eliana suggests, explain why!
132	Adin	It has more of a chance…
133	Teacher	So if it has more of a chance you are marking it on the 30, more chance for what?
134	Adin	More of a chance than A, B and C. There is a higher chance that it will happen, it is closer to the middle.
135	Teacher	So if there is a higher chance you are marking it close to what? Does anyone feel different, want to support or oppose? ... What do you think, Guy?
136	Guy	I think it is much higher [Teacher asks how high?] - 80%, because in fact there are 4 sides to the top, right? And the chance that it will fall on one of them is 25% and you said that it will fall between 20 to 30, so…
137	Yael	That means that it is 25% not 80%.
138	Guy	Not that it will fall on it 25 times, on it… out of 100… 80% approximately 90%.
139	Teacher	What do you say about what Guy says?
140	Guy	Just a second, can I continue? It is not how many times it fell, I can't explain.

141 Omri	What I am trying to see is if I understood Guy: what he is trying to say is that there is a 1 out of 4 chance, that means that it is a very high percentage that it will be between 20 to 30.
142 Teacher	Yes, that means you support Guy?
143 Omri	Yes!
144 Teacher	Can you explain again why you are supporting Guy?
145 Omri	What he's saying is that every time you spin there is a 1 out of 4 chance that it will fall on the N, meaning, 25% now out of 100 is approximately the number of times it will fall on the N, because it is ¼ out of four.
146 Teacher	What do you think? You are nodding yes (turns to Rachel); who do you agree with?
147 Rachel	With Guy.

Note that events C & D are different from events A, & B, which discussed repetition of the same simple event. C & D address the probability of falling into a range of values; this probability is determined not only by the size of that range but by its position relative to the expected value. Given that two ranges are of equal size, the one whose centre is closer to the expected value has a higher probability. **Era** is a simplified version of this statement; in the a priori analysis, we decided that this simplified version can be expected to be developed by Grade 8 students.

Adin (130, 132 & 134) tries to answer these questions, carried by an intuitive feeling that D has greater chances than events A, B & C. However, the probability he indicates is lower than the correct one and he can't suggest any explanation for the teacher's prompt question in 133. We do not consider Adins' sequence of claims as creative ideas, because although they are novel, they lack justifications. In turn 136 Guy suggests the chances are 80%, and also gives an appropriate explanation, by presenting a new claim followed by a justification, he expressed creative idea. We may conclude that Guy expresses the knowledge element **Era**. Does he start a new shift of this creative idea in the classroom? Not immediately! In turn 137 Yael follows only the first part of Guy's idea. Guy (138 & 140), who would like to explain again his reasoning is lacking words. Only then Omri (141 & 145) repeats Guy's entire creative reasoning in a very clear way, and by this he makes Guy a knowledge agent and himself a follower. An additional creative way of reasoning starts its shift in the classroom.

During the group work, which takes place just after this episode, we followed the group of Yael, Rachel and Noam. We bring only the beginning of the episode here.

Group work – Episode 3, Evidence for downloading of creative knowledge

The students discuss the Coin Problem (Figure 2).

174 Yael	[Reads the problem]. There are only two sides to a coin.
175 Rachel	But still…
176 Noam	It is not like the top.
177 Yael	In my opinion it is a quarter! [Yael marks a quarter on the chance bar].
…	
182 Rachel	If you are saying that it will be 1000 times tails, at the same time it has the same chance being heads.
183 Yael	It's like you throw a coin 10 times and it came out tails 5 times and 5 times heads, you can't say that with 1000 it will be 500 tails and 500 heads [Yael corrects the chance to lower, closer to 0].
…	
191 Yael	The more throws you add your chances are decreasing.
192 Rachel	But if there are only 2 sides, and if it doesn't fall on tails, and you yourself say that there isn't a high chance it will come out a 1000 times tails, so there are many times it will come out heads. It's actually what you are saying because a coin has two sides.
193 Yael	No, I am saying that there is a higher chance that it will come out tails rather than 100 times, 100 times N with the top…look, every time you throw one more throw, every time you throw more times the chances decreases.

From turn 174 to 190 the three girls in the group raise different ideas. Some of these ideas do not include any reasoning to which we can assign any meaning (e.g. Yael in 177). Others express the struggle with the situation (e.g. Rachel in 182 and Yael in 183). However, Yael's turn 191 (and again turns 207 and 218 – not shown here) demonstrates how ideas continue to shift, this time from the whole class discussion to the focus group: Quite surprisingly Yael, in turn 191, repeats the idea raised by Guy in a much earlier episode, Episode 1, turn 73: "The more throws you add, your chances are decreasing" and repeats it again in 193. We may conclude that in this episode Yael (1) downloads Guy's idea from the whole class to the group without mentioning his name, and (2) follows Guy and this provides somewhat delayed evidence that in Episode 1 Guy was a knowledge agent. In addition, Yael's contribution in 191 and more so in 193 might be considered as creative reasoning, in spite of being incomplete, since she adapts Guy's idea to a new mathematical situation.

DISCUSSION

Our general question in this paper is: What are the mutual relations between knowledge agents and their followers in the classroom and the creative ideas within mathematical knowledge shifts in the classroom? Our findings reveal shifts of creative knowledge in an inquiry based mathematics classroom. We showed one shift that took place within the whole class discussion, and one shift that "crossed the lines" and represents a downloading of a creative idea from the whole class into a working group. The shifts of creative knowledge started with an individual student who raised a creative idea that was new to the learners and often served as a kind of milestone in the inquiry of the topic. In what way is the shifted knowledge creative? We claim that the analysis of the above sequence of episodes fits our intuitive definition of creativity mentioned at the beginning of this paper, and also appropriate to Lithners' (2008) three criteria.

This study is, on the one hand, a continuation of our line of research on knowledge shifts between different classroom settings, research that is situated within the DCA and AiC frameworks and methodologies. On the other hand, this is our first attempt at expanding this line of research into creativity as it unfolds in the classroom.

Acknowledgement

This study was partially supported by Israel Science Foundation grants 1057/12 and 438/15.

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CHANGE IN TEACHERS' PRACTICES TOWARDS EXPLORATIVE INSTRUCTION

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Our study tracks change in teachers' practice in the context of a professional development (PD) program aimed at getting teachers to teach mathematics "exploratively": using cognitively demanding tasks, exploration phase in groups and whole-classroom discussions. Participants included 7 middle school teachers in an urban district in eastern United States. Our data shows variability in uptake of the PD principles, with an improvement in Accountable Talk measures when lesson plans were highly scaffolded for the teachers. The scaffolding was done by clear lesson plans and instruction about Accountable Talk moves. We focus on one particular case to show the qualitative differences in students' discussion over two lessons.

What does it take to transform mathematics instruction in challenging, urban settings? Despite evidence that robust, conceptual learning is significantly enhanced by discourse-rich, highly cognitive-demanding instruction, students that most often need this instruction still sit in classrooms where teaching is of the "demonstrate and practice" kind. This gap is probably a result of the fact that the change needed is not merely one of textbooks or of instructional protocol, but rather a shift in classroom culture. Teachers play a crucial role in supporting this shift. Understanding what it takes to move teachers towards mathematics instruction that supports explorative participation is the goal of the present research.

THEORETICAL BACKGROUND

Explorative participation in mathematical learning is defined by participation for the sake of producing mathematical narratives to solve problems or to describe the world (Heyd-Metzuyanim, 2015; Sfard, 2008). Such is the participation teachers and educators wish to cultivate in mathematics classrooms. Yet often enough, students are found to participate *ritually* in mathematical learning. Such ritual participation is geared at pleasing an authority, often the teacher, getting high grades, or simply being identified as a "good student" (Heyd-Metzuyanim & Graven, 2015).

Instruction that supports explorative mathematical learning is characterized by several features. It provides tasks that are cognitively demanding and are open to different solutions and procedures (Smith & Stein, 2011), thus minimizing the propensity for ritual memorization of rules and procedures. Explorative Instruction is organized around group work which promotes students' agency and authority (Boaler & Greeno, 2000); and it fosters discussions characterized by Accountable Talk, in which students are held accountable to each other and to rigorous reasoning (Bill, Leer, Reams, & Resnick, 1992; Resnick, Michaels, & O'Connor, 2010), thus diminishing the ritual tendency to seek solely the approval of the teacher.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 393–400. Szeged, Hungary: PME.

Over the past two decades, accumulating evidence has shown that this type of instruction promotes conceptual understanding, strengthens students' mathematical identities, and even transfers to higher achievements in other subject domains (Resnick, 2015; Schoenfeld, 2014). Yet despite these findings, mathematics classrooms, especially in urban settings, are still often dominated by instruction that promoted ritual participation (Jacobs et al., 2006).

In recent years, certain professional development programs have reported success in changing teachers' practice from ritual towards a more explorative-oriented instruction (e.g. Boston & Smith, 2009). Others have found such change difficult to accomplish, especially in urban settings (e.g. Clarke, Chen, Stainton, & Katz, 2013). Less attention has been given to the *process* by which teachers change their practice. In relation to this process, we apply Sfard's (2008) socio-cultural lens, together with its distinction between ritual and explorative participation. This lens provides a dual view on change in teachers' practice. On the first level, it points to the necessary changes in classroom discourse, including the opportunities given for students' participation in this discourse. On the second level, it points to teachers' appropriation of tools (such as tasks, classroom procedures and talk-moves during discussions) that enable the changes in classroom discourse. At both these levels, the distinction can be made between ritual and explorative participation. Just as students can participate ritually in mathematical learning, so can teachers participate ritually in practices promoted by a PD program. For instance, they can use a high-cognitively demanding task because it has been recommended (or even dictated) by the teacher educators, yet implement it in ways that are not consistent with the goals of the task, for instance by proceduralizing it (Henningsen & Stein, 1997). Thus, our aim in this study was to use the ritual-explorative framework to investigate *the process* of teachers' change towards discourse-rich, cognitively demanding instruction.

The Intervention: A Professional Development and Coaching Program

Our research was located in an urban district in eastern United States. The professional development (PD) was carried out under the organizational umbrella of the Institute For Learning (http://ifl.pitt.edu) and was led by Margaret Smith and Victoria Bill. In a nutshell, the PD program featured the following characteristics: 1. Teachers were trained in the "5 practices for orchestrating productive mathematics discussions" (Smith & Stein, 2011), which are practices for selecting high cognitive demand tasks and leading discussions about them in purposeful, planned ways. 2. Teachers were trained to use Accountable Talk moves to promote students' reasoning, mathematical justifications and listening to each other (Resnick et al., 2010). 3. Training was situated in the actual life and work of teachers. (Resnick & Glennan, 2002). And 4. Teacher leaders were trained as coaches so that they can support teachers throughout the year and beyond the PD (West & Staub, 2003). In the context of this PD, the question of our research was: to what extent do teachers change their instructional practices in the classroom and if so, how can this change be characterized?

METHOD

The study took advantage of an already-established PD program for 50 middle-school teachers and 11 teacher-leaders, which took place between August 2014 – March 2015. In addition to 4 teacher training sessions there were 4 additional sessions for teacher leaders. Throughout the year, teachers were supported via individual coaching sessions by the teacher leaders and via a web-based platform provided by LearnZillion. 7 teachers and 5 teacher leaders, a subset of the teachers and coaches participating in the PD, volunteered to participate in the study. They were recruited during the first PD session in August and then followed until May 2015.

The data collection included four cycles, each containing: 1. Pre-lesson interview with the teacher, about his/her plan for the lesson; 2. Video recording of a lesson planned by the teachers according to the "5 Practices". 3. Students' worksheets produced during the lesson. 4. Post-lesson interview with the teacher, reflecting upon the lesson.

In addition, all PD sessions were recorded, interviews were held with the coaches during and at the end of the PD program, and interviews were held with each of the teachers at the beginning and end of the year.

Data Analysis. For examining change in teachers practice we used a three-tiered analysis design:

Tier 1: Measuring the potential of the tasks used in the lessons was done with the Instructional Quality Assessment tool (IQA) (Boston & Smith, 2009), which provides a score of 1-4 on the cognitive demand of the task. A score of 1 means the task only demands rote memorization, 2 means the task only invites the application of procedures explicitly taught, 3 is a high level task that is flawed in some way (e.g. doesn't invite verbal explanations) and 4 is a high-level task which invites engagement with mathematical ideas and does not have only one, procedural way for solving it.

Tier 2: To determine quantitate changes in teachers' talk during whole-classroom discussions, we used the Accountable Talk (AT) coding scheme (Clarke et al., 2013). This scheme, which we slightly modified for our study's purposes, codes classroom transcription on a line-by-line basis. Codes for teachers' talk are: Press for Reasoning ("why?", "how do you see that?", Challenge ("But isn't it…?"), Agree/Disagree ("Who agrees with what Daniel says?"), Add-On ("who wants to add on to what Jayla has said?"), Say More ("can you elaborate"?), Revoice ("What I'm hearing you say is…", Explain other/Restate ("who wants to explain Tom's idea in his own words"?).

Tier 3: Excerpts that have been shown to differ significantly in quantitative measures of Accountable Talk (as produced by the Tier 2 analysis) were examined qualitatively using Sfard's communicational framework (Sfard, 2008) to determine the opportunities offered for students' participation. In particular, we looked for change in authority structure and in who initiates ideas, the place given for wrong answers, and opportunities for mathematical justification, all parts of explorative participation.

FINDINGS

The coding revealed perceptible changes in teachers AT moves. Figure 1a shows the total count of teachers' AT moves during whole classroom discussions. Each line stands for one teacher and traces the changes in total AT teacher moves during the 4 lessons (notice the line graph is used to show trends, not to imply continuity). The graph shows AT peaked in most lessons (4 out of 7) during the 3rd lesson and then dropped down during the last follow-up (4th lesson).

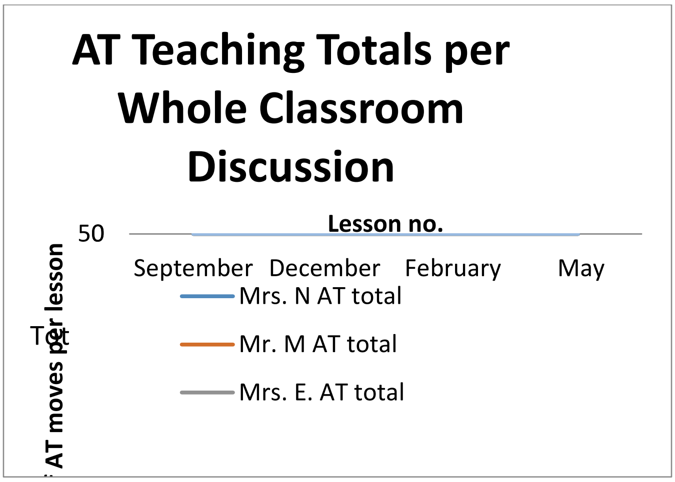

Figure 1a

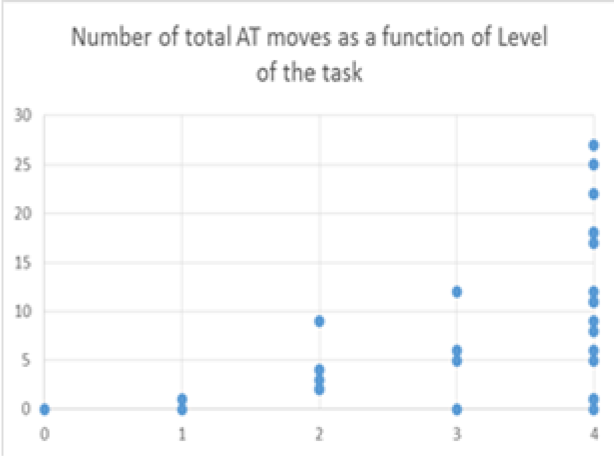

Figure 2b

Regarding the cognitive level of the task, Cross-checking the AT counts with IQA "potential of the task" scores revealed a clear relationship (see fig 1b), meaning cognitive level of the task seemed to be a *necessary but not sufficient* condition for high levels of AT. *Necessary*, because when the level of the task was low (1 or 2), almost all lessons had less than 5 AT moves. *Not sufficient*, because even when the level was the highest (4), six lessons still remained with a low-to-moderate AT score (less than 10). This finding corroborates ealier studies (Henningsen & Stein, 1997) that pointed to the importance and the "ceiling effect" of the level of the task. The novelity here is that this was obtained by two different measures; one pretaining to the cognitive demand of the task as written, the other to measures of teacher talk during classroom discussion.

The importance of the potential of the task was further enhanced by the findings showing a peak in AT talk during the 3rd lesson in 4 out of the 6 classrooms. During this lesson, all teachers participating in the PD were asked to implement the same task, namely the "Hexagon Task", a task which invites multiple algebraic solutions for calculating the perimeter of a series of "trains" made up of adjacent hexagons. Before the lesson, teachers collectively prepared a "monitoring sheet" for anticipating students' possible solutions, and were asked to write specific questions for pressing on students' reasoning for the different solutions. Finally, they were introduced to Accountable Talk moves that can facilitate the lesson discussion. Though this lesson was successful and important for almost all teachers invovled, for 3 teachers (Mr. D, Ms. N and Mr. M), this close scaffolding of the lesson plan proved crucially important.

The amount of AT moves in their lessons rose considerably during these lessons (see Fig 1). In fact, their graphs show that they were unable to sustain this high level of AT in their 4th lesson. This finding suggests these teachers were in a ritual stage of implementing the PD practices, and that they may have needed to implement several more scaffolded lessons before they would be able to choose and implement tasks with high levels of AT discussion.

Out of the whole group of 7 teachers, one pair of teachers – Ms. M & Ms. W., proved to change their instructional discourse most considerably. In what follows, we briefly present excerpts from their first and second lessons, where we found the change to be most perceptible both in AT and in qualitative terms.

Ms. M & Ms. W co-taught an inclusive 6th grade classroom at one of the lowest-achieving schools in the district. Ms. M was the principal teacher while Ms. W. was a special education teacher. Unlike similar teacher teams, they chose to plan lessons together and co-teach them instead of Ms. W working with the group of special needs students separately. They had several years of experience working in such a way and the co-teaching seemed to be especially productive for their taking up of the PD principles. When they started with the "5 practices" at the beginning of the year, Ms. M had already had some knowledge and experience with talk moves that encourage discussion. On the first lesson, Ms. M and Ms. W presented the following problem:

> A publishing company is looking for new employees to type novels that will soon be published. The publishing company wants to find someone who can type at least 45 words per minute. Dominique discovered she can type at a constant rate of 704 words in 16 minutes. Does Dominique type fast enough to qualify for the job? Explain why or why not.

The task was graded by the IQA as a high-level (4) task because: a. at that point, students did not have well-practiced procedures for solving it. B. There were multiple solutions paths possible (division of 704:16; multiplication of 45*16; working with a "ratio table", and more).

During the "explore" phase of the lesson (when students were working in groups), some students were struggling with the task, others found some solutions. However, when starting the whole-classroom discussion part of the lesson, Ms. M & Ms. W chose to invite only the students that had correct solutions. These students presented their work rather hesitantly, and most of the explanation was "restated" by the teachers. For instance, when Justin presented his ratio table, the following interaction occurred:

Ms. M	A ratio table - okay, Justin - …- can you explain to me what your thinking was when you were creating your ratio table?
Justin	Uh, Dominique did 45 minutes in words, and you gotta see how many times she did it, and - and, um...
Ms. W	Wait, hold on right there. So I - she - the job says that you have to type at least 45 words per minute, right? Okay, so you were <u>looking</u> to find out if Dominique could type fast enough. So, explain how this table works.

Justin	In - in her [Indistinct] 60 seconds I - I did, I counted one all the way to 16, and [I meet] 720.
T.W	Oh. So you found out, [if I'm correct], that you need to be able to type 720 words in 16 minutes to get the job.

This excerpt, though very short, is generally indicative of the authority structure during the 1st lesson. Though students were invited to present their solutions, the role of clarifying their mathematical thoughts remained solely on the shoulders of the teachers. No one held Justin accountable to explain his thinking more clearly, as could be seen in both Ms. W and Ms. M's eagerness to restate his confused words into statements that could be understandable by the rest of the students. This was also the case with the next student who presented a solution path involving division, though that student was more articulate, and therefore needed less "restating". In general, the teachers seemed to use student presentations in this lesson as a proxy for explaining the different solution paths in the ways they had wished them to be presented. Though students' presentations of different solution paths are an important part of the "5 practices" plan, such an implementation keeps the lesson at a level of "show and tell" and does not offer students real agency and authority for grappling with mathematical ideas. This authority structure changed remarkably in the 2nd lesson. For the 2nd lesson, the students were given the following task:

> Christian, Cayden and Annabella were playing a card game over vacation. The object of the game is to finish with the most points. The scores at the end of the game are: Christian -1, Cayden -2, and Annabella -4. Who won the game?

This time, the teachers chose first to bring up to the board a student who made an erroneous claim, but was able to convince his fellow group members that Annabella (with -4 points) was the one to win the game. When presenting his work on the board, he explained: "Even though 1 is closer to the 0, it's still bigger than ... it's still ... 4 is bigger than 1, even though it's farther away from the zero." In reaction to this claim, a genuine discussion developed, in which several students questioned Roger's claim. Dawson asked "wouldn't it be Christian, 'cause 1 is closer to the 0?" and Jayla, who had formerly been convinced by Roger's arguments during the group discussion, had a sudden 'a-ha moment'. Urged by the teachers to share it with the rest of the class, she explained, pointing to the -4 on the number-line at the board: "four- left is little, and right is greater than. So the 4 is smaller than the 1". Pressed by the teachers for the meaning of "four", she corrected herself to "it's a negative four". Throughout the whole discussion, students were visibly more engaged in the discussion than during the first lesson. They responded to each other's suggestions, and built on previous knowledge. For instance, Andrew, explaining why he agreed with the idea that Christian (-1) was the winner, explained "Because, the other day we talked about - if you want to owe Ms. W - whether you wanted to owe Ms. W 4 dollars or 1 dollar? … And if you owe 4 dollars, you're losing 4 dollars from yourself". Thus the authority structure in this lesson was markedly different than the 1st lesson. Instead of students presenting their solutions merely for the sake of showcasing the teachers' narratives (a modified version of the “show and tell” routine), students were now having agency to err, disagree, argue with each other and change their minds about mathematical ideas. We hypothesize that this movement in discourse structure was at least partially a result of several trial-and-error attempts at introducing such tasks to the classroom. Ms. M & Ms. W, though starting already at a relatively high level of implementation of the “5 practices” principles, still did not have enough experience during the first lesson to enable genuine student discussion. Their level of expertise continued to rise, seen in their AT in lessons 3 and 4, and most notably in a follow up lesson they recorded themselves, which included again more than 20 AT moves (in a whole-classroom discussion of about 15 minutes).

SUMMARY

In the present study, we sought to examine teachers’ change in instructional practices through two measures, IQA level of task and Accountable Talk during whole-classroom discussions. AT results showed great variability in AT talk between teachers as well as variability in choosing cognitively demanding tasks. Yet overall, there seemed to be a positive change, especially when the lesson-plan was well scaffolded and where the task was sufficiently rich and appropriate for a variety of classrooms. Implications of this study are relevant both for researchers and for professional development practitioners. They support previous studies stating change in teachers’ instructional practice is a difficult and non-linear process that may take more than a year (Gresalfi & Cobb, 2011). They also show that a “try first based on scaffolded teacher materials and “design independently later” may be a productive approach for initiating teachers into such complex changes in their practice, as some complexities to this type of instruction cannot be mastered until some experience had been gained with it.

Acknowledgments

The research reported here was supported by Spencer Foundation, grant 201500080.

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DEVELOPMENT OF A REAL TEACHING SITUATION-BASED MEASURE OF MATHEMATICS TEACHING COMPETENCE IN TAIWAN

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In TEDS-M, an international comparison study sponsored by the IEA, numerous MPCK items were considered as measuring MCK or as involving situations that would not occur in Taiwan. The present study was the first attempt in Taiwan to develop a conceptual framework and test items for assessing preservice secondary mathematics teachers' mathematics teaching competence. The test items developed in this study were all based on situations that occurred in Taiwan classrooms and were obtained through observing mathematics instruction of 47 preservice mathematics teachers. A questionnaire composed of TEDS-M MPCK items and our items was developed, and these items were tested by 35 preservice teachers. The results showed that the preservice teachers performed much less well in Taiwan items than in TEDS-M items.

INTRODUCTION

Hill, Ball, and Schilling (2008) asserted that conceptualizing *pedagogical content knowledge* (PCK), a term that was coined by Shulman (1987), is still in the initial stage. Hsieh (2013) conducted a literature review and claimed that the development of conceptual frameworks for measuring mathematics PCK (MPCK) has been hindered because all of the MPCK models are either descriptions of certain ideas or are constructs that use broad and undetailed categories, and almost all of these models are based on Western views.

Cross countries' teachers' MPCK have been compared through the *Teacher Education and Development Study in Mathematics (TEDS-M*), an international comparison study of preservice mathematics teachers sponsored by the International Association for the Evaluation of Educational Achievement (Tatto et al., 2008), and the Mathematics Teaching in the 21st Century study (MT21, Schmidt et al., 2011). The first author of the present study, as the national research coordinator for Taiwan in these two studies, discovered an eager to develop more conceptual frameworks and test items, especially for countries with non-Western educational cultures, for several reasons, including the following: knowledge types are not subdivided into specifically itemized topics in these studies, and the situations used in the MPCK items in the TEDS-M are too simple or do not accurately represent real mathematics classrooms at secondary schools in Taiwan. Hsieh (2009) proposed a theoretical framework for mathematics teaching competence (MTC). She used the term *competence* rather than *knowledge* to represent the skills, knowledge, qualifications, and capacity related to the cohesive unit of pedagogy and mathematics in the teaching context (Hsieh, 2013). Her model may include an "endless" list of competences. Thus, for a study with a limited scope, a conceptual framework specific to the purpose of the study is required, for which Hsieh's model may be referenced. The present study assumed that the main aim of teacher preparation is to cultivate teachers who can provide effective instruction. Developing measures for evaluating competence assemble to a practical teaching context should be a goal. Based on these premises, the present study had two goals:

(1) To explore the conceptual framework and test items that are specific to the framework for measuring the real teaching situation-based MTC.

(2) To investigate the MTC of Taiwan preservice secondary mathematics teachers by using conceptual frameworks that are integrated with the frameworks developed in (1).

LITERATURE REVIEW

Several projects and scholars have developed measures for assessing the knowledge component of teacher MTC, but different terms have been used, such as MPCK in the TEDS-M, *mathematics pedagogy knowledge* in the MT21, and *mathematical knowledge for teaching* (MKT) in the study by Hill, Schilling, and Ball (2004); others, such as Ernest (1989) from the United Kingdom and Krauss et al. (2008) from Germany, have used only general terms. Pepin (1999) reviewed and compared existing models for teaching in Anglo/American, French, and German settings and concluded that all the models are mere conceptual descriptions or include constructs that only list a few broad categories of knowledge.

In the United States, Ball, Hill, and their colleagues, have delineated PCK in mathematics as being composed of knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC; Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). Hill, Ball, and Schilling (2008) further developed and tested the items specific to their model. KCS combines the knowledge of mathematics and that of students, including anticipating and interpreting student thinking and predicting what information is difficult or easy for students to comprehend. KCT combines the understanding of mathematics and that of pedagogy. Thus, KCT is pertinent to various aspects of teaching, such as teachers' sequencing of content for instruction, choosing an introductory example for teaching a specific topic, and evaluating the advantages and disadvantages of using certain representations to teach a topic. KCC involves teachers' comprehension of how topics are arranged and connected and the advantages, limitations of various curriculum design, etc. (Hill, Schilling, & Ball, 2004).

The first author of the present study spent 4 years developing frameworks and indicators of MTC by conducting a series of studies in Taiwan (Hsieh, 2006, 2009, 2012). Hsieh's model of MTC frameworks is centered on three objects: *elements*, *operations*, and *kernels*. Hsieh (2013) identified 20 elements, such as mathematics representation, mathematics teaching method, mathematics thinking, etc. The elements would be engaged by three operations: *recognizing and understanding*, *thinking and reasoning* (TR), and *conceptual executing* (CE). The focus of the MTCs can be directed through the three kernels of perspective: *learning*, *teaching*, and *entity*. For example, using the TR operation to engage the element *mathematics language* with the *entity* kernel as a focus, a MTC may be as follows: Being able to distinguish the features specific to mathematics language that are not inherent in daily life.

METHODOLOGY

Study Subjects

One type of subject in this study was the objects: the conceptual framework for real teaching situation-based MTC and the test items specific to that framework. Another type of subject in this study was the preservice secondary mathematics teachers. The sample comprised 47 preservice mathematics teachers who were enrolled in teaching practice courses during their fourth year of undergraduate studies in the 2013 and 2014 school years and 35 preservice teachers who were enrolled in school-based teaching practicums in 2014.

Design and Instruments

Hsieh's theoretical frameworks of MTC (Hsieh, 2009, 2013) and Hill, Ball, and Schilling's (2008) MKT model were used as a blueprint for developing the conceptual frameworks and test items for real teaching situation-based MTC. The research methods used were a literature review, observations of peer and field teaching, video analyses, and a focus group discussion, which involved seven experts, including researchers and secondary mathematics teachers with an average of 8.7 years of teaching experience. A total of 94 peer and field instructional sessions of the sample, who were enrolled in teaching practicum, were observed and videotaped. In this initial stage, approximately one-third of the videos were analyzed and used in this study. The experts identified typical and controversial teaching segments, discussed the MTC that is required to address the problems in the segments for developing frameworks, and used the segments to construct the test items, hereafter referred to as the Taiwan MTC items. The situations described in the items directly match situations that occurred during the observed mathematics instruction.

To investigate preservice teacher MTC, both the TEDS-M MPCK and Taiwan MTC items were used form a questionnaire. Among the 29 TEDS-M MPCK items, only 16 were regarded as measuring MPCK by the Taiwan team, all of which were included in the questionnaire. Thus, for Taiwan MTC items, 9 teaching segments were chosen to form 16 test items, among which 9 were open-ended and 7 were multiple-choice questions. The questionnaire was employed to survey 35 preservice teachers in teaching practicums.

Data Analysis

Scoring sessions were conducted to develop scoring rubrics through content and inductive analyses and to assign scores to responses to open-ended items. There were nine scorers: two professors, one Ph.D. student, and six secondary mathematics teachers with master's degrees. Each item was scored by two scorers; when their scores did not match, they consulted each other to reach a final decision.

The quality of the test items was evaluated according to item difficulty (indexed with *p*) and item discrimination. An item was classified as easy if $p \geqq 0.7$, moderate if $0.7 \geqq p \geqq 0.3$, and hard if $p \leqq 0.3$ (Ahmananm & Glock, 1981). An item was classified as having excellent, good, acceptable, and poor item discriminations if $D \geqq 0.4$, $0.4 > D \geqq 0.3$, $0.3 > D \geqq 0.2$, and $D < 0.2$, respectively (Ebel & Frisbie, 1991). The percent corrects for each element in the Taiwan MTC framework and the categories with combined elements in this framework (Table 1) were respectively calculated and compared using paired *t* tests.

RESULTS

Conceptual Framework and Measures of Real Teaching Situation-Based MTC

The present study found that the inadequate teaching of Taiwan preservice teachers was manifested in various facets. The real teaching situation-based MTC frameworks and the developed Taiwan MTC items were classified into the aforementioned models, as displayed in Table 1.

MKT model	Hsieh's model			
		Taiwan MTC items		TEDS-M items
		Operations		
	Elements	TR	CE	TR
KCT	Math teaching process (TP)	6	1	
	Math teaching method (TD)		1	
	Math teaching material (TM)	2	1	5
	Math language (L)	2	1	
	Math representation (R)	4		
	Math evaluation (E)	1		3
KCS	Math cognition and understanding (CU)	2	1	
	Math competence (C)	4		1
	Math thinking (T)	1		
	Math problem solving (PS)	1		
	Math misconceptions (M)	1		4
KCC	Math curriculum (CR)			3
	Total	16		16

Table 1: Framework and measures of MTC integrated into the context of Taiwan

The TEDS-M items served as a reference for analyzing the Taiwan MTC items. Regarding item difficulty (Figure 1), most of the Taiwan MTC items (69%) were at the moderate level, but most of the TEDS-M items (75%) were at the easy level. Regarding item discrimination, more than half of the Taiwan MTC items (56%) were at excellent, good, or acceptable levels, but most of the TEDS-M items (69%) were below the acceptable level; this result was attributed to the high percent corrects (89%–100%) for these items (except two). For all the items that were determined to have poor discrimination, Kruskal–Wallis tests were conducted to compare the percent corrects among the three groups (the overall test scores ranged at top 25%, middle 50%, and bottom 25%). The results showed that all the items, except one TEDS-M item, did not yield significantly different percent corrects for the three groups.

MTC of Taiwan Preservice Secondary Mathematics Teachers

The average percent corrects of overall test, Taiwan MTC items, and TEDS-M items were 68%, 53%, and 83%, respectively. The average percent correct for the TEDS-M items in our sample did not significantly differ from that in the Taiwan sample tested in 2008 (82%), but significantly exceeded the international average (52%).

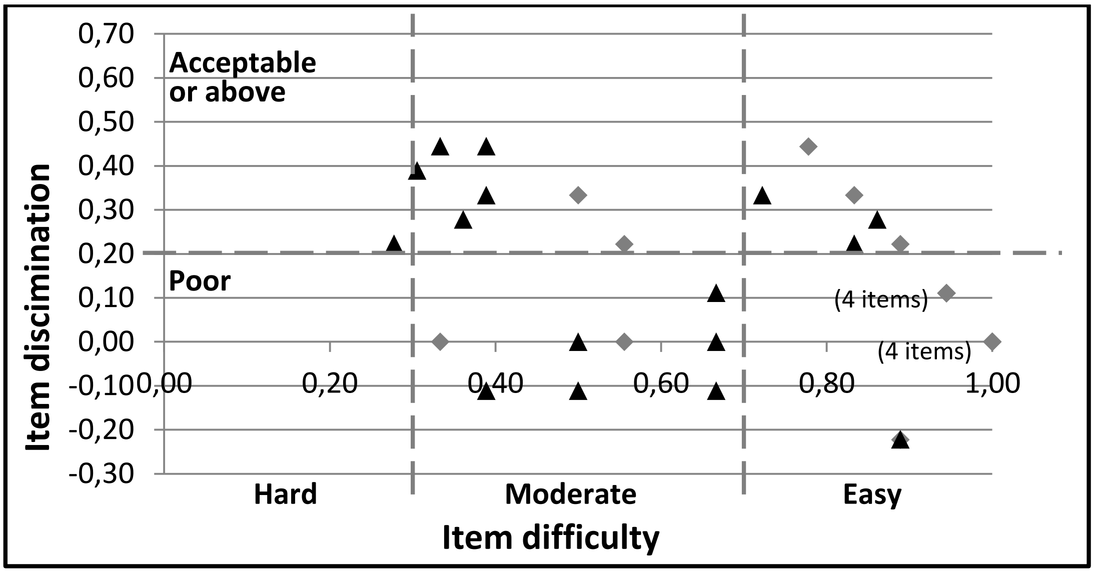


Figure 1: Results of item analysis. The triangular marks indicate the Taiwan MTC items, and the rhombic marks indicate the TEDS-M items.

To highlight real teaching situation-based MTC, only Taiwan MTC items are reported here. Moreover, the elements with more closed concepts were combined to form six categories to increase the number of items in each category, as displayed in Table 1. The findings (Figure 2) showed that our subjects performed the most highly in the categories of language and representation and student misconceptions, both with an average percent correct was 60%. The weakest category for the participants was managing teaching materials and homework; the average percent correct for this category was 36%.

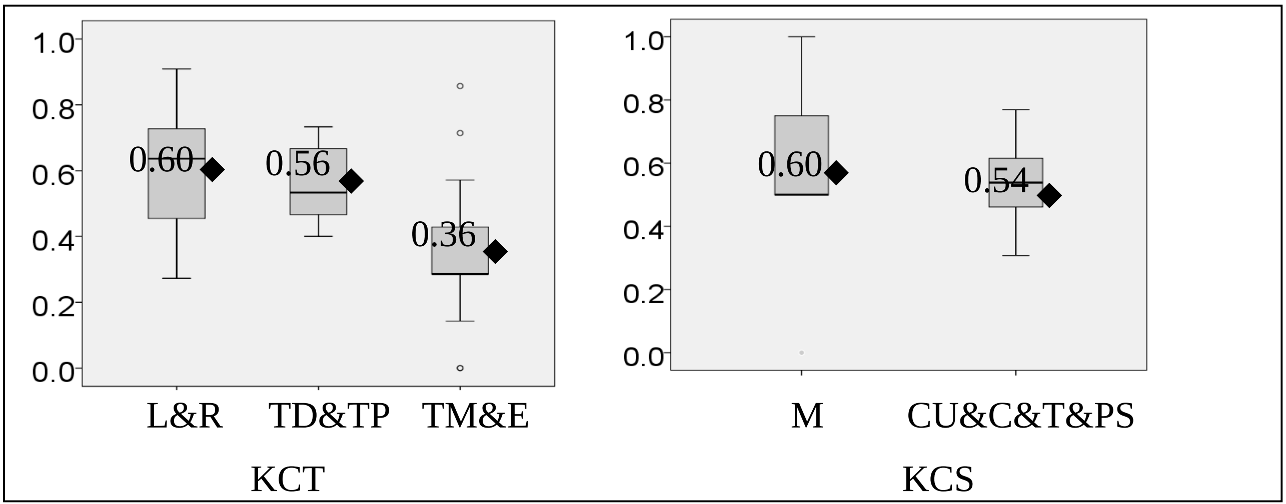


Figure 2: Box plots of percent corrects of Taiwan MTC items. The numbers are the average precent corrects. See Table 1 for a description of the abbreviations.

To illustrate the Taiwan MTC items and the performance of the participating preservice teachers, item MK7 is reported in this paper (Table 2). This item was used to investigate the preservice teachers' competence in judging whether a problem is suitable for homework according to the classroom teaching context. The correct answer for this item was (4) because the students were recently taught the Pythagorean theorem; a continued ratio was not easy for the students to connect to this theorem without teacher instruction. Additionally, an item that requires calculation and deep thinking should not be provided in a true–false format.

The fact that only 42.9% of the preservice teachers checked (4) illustrated that they did not master the ability to judge the suitability of the first homework assignment for a newly introduced concept. A total of 45.7% of the preservice teachers checked (2). These preservice teachers may have considered (2) as relating to understanding mathematics terminology rather than applying $a^2 + b^2 = c^2$. However, (2) is appropriate in a true–false format for novice learners because Taiwan teachers often use the terms "leg" and "hypotenuse" in mathematics classes; understanding these terms is a basic requirement for further learning.

Conclusion

The Taiwan, MTC items are much more acceptable than the TEDS-M items regarding item difficulty and discrimination, and item development is still ongoing. It has been shown that Taiwan preservice secondary mathematics teachers ranked at the first place in TEDS-M MPCK items; our subjects, who had similar competence levels, did not perform highly on the Taiwan MTC items, which were developed according to real teaching contexts. Whether these preservice teachers can perform effectively in real Taiwan mathematics classes still merits investigation. The present study was the first to develop a conceptual framework for real teaching situation-based MTC and test items that are specific to this framework; this framework is still in the initial stage, and more analyses should be conducted to complete it.

MK7	Element: Mathematics evaluation	Operation: TR

Mr. Ho used an area exploration activity to introduce the Pythagorean theorem to his students. He then began to show the students how to solve Example 1, which is displayed in the following figure, by orally dictating the following four problem-solving steps:

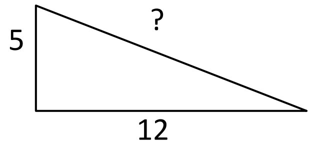
(1) "The problem asks for the length of a side. Once one sees a question on the length of a side, one must suppose an unknown"

(2) "Suppose the hypotenuse is c; the length of the three sides can be represented by (5,12, *c*)."

(3) "According to the Pythagorean theorem, we can obtain $5^2 + 12^2 = c^2$."

(4) "5^2 plus 12^2 equals 169. Therefore, *c* equals ±13; thus, the length of the hypotenuse is 13."

Example 1: How long is the hypotenuse? (unit: cm)



After solving Example 1, Mr. Ho solved three more examples and asked the students to solve four similar exercises. In all the problems, a right triangle and the lengths of two of its sides were provided, and the students were asked to determine the length of the third side. After all the problems were solved, the bell rang for class dismissal. Mr. Ho then assigned homework for the students. One of the problems was a true-false item which is displayed as follows:

For each of the following statements, mark "O" if it is correct and "X" if it is incorrect:
( ) (1) 26, 24, 10 can be the lengths of the three sides of a right triangle.
( ) (2) The longest side of a right triangle is called a leg.
( ) (3) If the lengths of the two legs of a right triangle are 3 and 3, then the length of the hypotenuse is shorter than 3.
( ) (4) The continued ratio of the three sides of an isosceles right triangle is 1:1:2.

For the aforementioned true–false item, please check the item that you think is the most inappropriate for homework.

□ (1) □ (2) □ (3) □ (4)

Percentages of respondents that checked the aforementioned items

(1) 8.6% (2) 45.7% (3) 2.9% (4) 42.9%

Table 2: Preservice teacher responses to item MK7

Acknowledgement

We gratefully acknowledge the following: the IEA, the ISC at MSU, the DPC, the ACER, the US NSF, and all TEDS-M national research coordinators for sponsoring the international study and providing information and data.

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ENGINEERING STUDENTS' USE OF INTUITION TO DECIDE ON THE VALIDITY OF MATHEMATICAL STATEMENTS

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This study explored engineering students' approaches to mathematical statements with unknown truth values. Task-based interviews utilizing the think-aloud method revealed students' reasoning processes in depth. The students in this study used three distinct types of intuitive reasoning to decide the truth value of mathematical statements. The results of this study indicate that if the constructed intuitive representation accurately represents task structures, such related interpretations will have a positive effect on reasoning, but if intuitive representations are distorted or deficient, they may lead to negative effects on reasoning.

INTRODUCTION

Many educators believe that students' desire for proof will be stimulated by opportunities to explore the truth value of mathematical statements. "A main challenge in teaching argumentation and proof is to motivate students to examine whether and why statements are true or false" (Durand-Guerrier et al., 2012, p. 362). Unfortunately, in the standard process of mathematics teaching, students are seldom required to construct proofs of unknown statements or to determine the truth value of mathematical statements (de Villiers, 2010; Durand-Guerrier et al., 2012). Because of the emphasis on syntactic reasoning and *prove this* statements in the undergraduate curriculum (Weber & Alock, 2004), little is known about how engineering students approach mathematical statements with unknown truth values.

Intuition is particularly important for determining the truth value of a mathematical statement, because in the absence of proof, it provides possibilities that students can then test (Burton, 2004; Fischbein, 1994). This study attempts to explore the use of intuitive reasoning and what types of systematic errors may inhibit success in the proving process during the processes of deciding on the truth value of mathematical statements by engineering students in an interview setting.

THEORETICAL FRAMEWORK

The intuition proposed by Fischbein (1982) is "a representation, an explanation or an interpretation directly accepted by us as something natural, self-evident, intrinsically meaningful, like a simple, given fact" (p. 10). Intuition takes into consideration the target of reasoning in prior knowledge, experience, conviction, task characteristics, and the creation of task representation (Evans, 2010). Furthermore, "intuition is able to organize information, to synthesize previously acquired experiences . . . to guess, by extrapolation, beyond the facts at hand" (Fischbein, 1982, p. 12). Organizing information intuitively provides a preliminary understanding of mathematical tasks

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 409–416. Szeged, Hungary: PME.

which can provide a starting point and suggest a direction can be followed (Burton, 2004; Fischbein, 1982, 1987). Intuitive representation in mathematics may be either a visual image or a perceptual representation of a concept or object (Tall, 2008).

Fischbein (1987) pointed out intuition is neither a source nor a method; it is a form of cognition. Unlike analytical thinking, it is a holistic leap of cognition. In his view, experience plays a crucial role in developing intuition. On the basis of stable and consistent experience, a thinker may learn to rely upon intuition, and it is quite autonomous in special circumstances. It may also impact individual judgment. Fischbein's (1987) classification is designed to clarify the complicated areas of intuitive cognition into two main types. The first type is classified according to the roles played by intuition (affirmatory, conjectural, anticipatory, and conclusive), and the second type by the origin of intuition (primary and secondary). Fischbein (1999) distinguished between affirmatory intuitions, which he described as direct and self-evident cognition without the need for checking or proving, and anticipatory intuitions, a sense of intrinsic conviction of one's ideas without any extrinsic encouragement.

Intuition is based on mental representations of tasks constructed from the clues given in a task and from the information retrieved from memory (Glockner & Witteman, 2010). This production of representations makes intuition significant in decision-making (Fischbein, 1987). Due to the inconsistency and incorrectness of previous learning experiences, the intuitive representation of individuals may not be able to authentically present the situation at hand. The reliability of intuition often depends on how intuition develops through related experiences (Burton, 2004; Evans, 2010). Many intuitive errors can be categorized as accessibility errors (Glockner & Witteman, 2010). *Accessibility* is the ease with which certain knowledge is evoked or certain task features are perceived and is a crucial component of intuitive reasoning and decision-making. There are two main types of accessibility errors, namely (1) attribute substitution, and (2) knowledge and task feature relevance.

Method

This research has interpretive approaches (Cohen et al., 2000, p.22). Case study is used as a research strategy to make an in-depth examination of students' intuitive reasoning in this study (ibid., p.181-182).The 23 first-year engineering students who participated in this study were enrolled at a university of technology and had learned the concepts of derivative and integration. This study explores the results of this process among engineering students rather than mathematicians, a choice more likely to produce values in teaching and "suggest learning trajectories that might be applicable for many other students as well" (Weber, 2009, p. 201).The mathematical task in this paper included two wrong statements regarding the concepts of differentiation and integration. For students, they were neither completely routine problems nor completely non-routine ones. The tasks refer to general objects and their properties and should be amenable to intuitive reaction. Participants completed the tasks in which they were asked to determine the truth value of the given mathematical statements and prove or disprove the statement accordingly.

Statement 1: If $\int_a^b f(x)dx \geq \int_a^b g(x)dx$, then$f(x) \geq g(x), \forall x \in [a,b]$.

Statement 2: If f(x) and $g(x)$ are all differentiable and f$'(x) > g'(x)$, $\forall x \in (a,b)$, then $f(x) > g(x), \forall x \in [a,b],, \forall x \in (a,b)$.

The data generated from (a) transcripts from the participants' task-based interviews using the think-aloud method, (b) participants' written work on the tasks in the interviews, and (c) my field notes from the interviews were categorized and coded (Miles & Huberman, 1984). As the process evolved, continuous comparisons were made between each category and the emerging new categories.

EMPIRICAL DATA AND ANALYSIS

The students used three distinct types of intuitive reasoning to decide the truth value of mathematical statements.

Logic-based Intuitive Reasoning

Logic-based intuition was the first type of intuitive reasoning used by students. It occurred only when determining the truth value of Statement 1. Three students made a logical mistake when determining the truth value of Statement 1. They intuitively believed that the mathematical statement and its converse are equivalent. For instance, the converse that S4 made in judging Statement 1 was correct, and hence Statement 1 is correct.

S4: This statement is apparently correct. Comparing f(x) = x2 + 1 and g(x)= x2, the value of f(x) is greater. The integral of f(x) from 0 to 1 is 4/3, while the integral of g(x) from 0 to 1 is 1/3. The integral of f(x) is also greater.

I: Did you find out f(x) and g(x) in the first place and that f(x) is greater than g(x), and then figured out that the integral of f(x) is greater than that of g(x).

S4: That's right! The greater the function is, the greater the integral will be.

I: Statement 1 says the integral is relatively greater, does this mean that the function is greater, too? The example that you provided just now indicates that the greater the function is, the greater the integral will be.

S4: Indeed. They work in the same way. If the function becomes greater, the integral will be greater as well, and vice versa.

According to Fischbein (1987), we can confirm that the equivalence of a statement and its converse (error) is a kind of intuition. Is it true that these three students did not possess formal logical schemas? Apparently this was not the case.

I: If f(x) is differentiable, will f(x) be continuous?

S4: Yes, f(x) is differentiable, so f(x) is continuous.

I: If f(x) is continuous, will f(x) be differentiable?

S4: If f(x) is continuous, I am not sure whether f(x) is differentiable. It is certain that if f(x) is discontinuous, then it could not be differentiated.

Apparently, a sufficient condition for determining the truth value of mathematics does not rely on whether the logical rules are well understood, and the student did not correctly apply logical rules to other scenarios. This supports the argument by Fischbein (1999), that intuitions are not absolute, they depend on the context. We can interpret these phenomena by describing two convictions (Fischbein, 1982) that may coexist: The first one is intuitive conviction, which means that a statement and its converse are equivalent; the second one is non-intuitive conviction, which means a statement and its contrapositive are equivalent. These three students like S4 could write out the statement that contrapositive is equivalence when elaborating on the relationship between differentiation and continuity. However, when determining the truth value of mathematical statement 1, more of their intuitive conviction came into play; they also confirmed that a statement and its converse are equivalent. In Fischbein's words, the first one was an *intuitive intrinsic type of conviction*, and the second one was a *formal extrinsic type of conviction*. In our view, the latter seems to have no impact on the former, which continued to be an obstacle.

Property-based Intuitive Reasoning

The second type of intuitive reasoning students used was property-based intuition. Students in this subgroup drew quick conclusions about the truth value of mathematical statements by using diagrams to represent "prototypical" examples of such mathematical statements. When confronting Statement 1, the property that students immediately thought of was area, and subsequently they directly used region areas surrounded by functional graphs to decide on the truth value of mathematical statements. Generally speaking, the students produced two different types of visual representation according to their intuitive representations. The first type of visual representation is one in which the diagram of functions f and g is located on two non-intersecting curves above the x axis. However, such diagrams may lead to wrong conclusions. Taking S6 as an example:

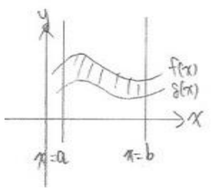
S6: This statement is apparently correct. This is because whenever I see integral, area comes to my mind. The integral value represents area, so the greater the integral value is, the greater the area will be, just like this figure I drew (Figure 1a). If the graph of f is here, then the graph of g will have to be drawn in this way, so that area will be bigger, and the function value of f will also be greater than that of g. Therefore, this statement is correct.

I: The functional graphs you have drawn are all above the x axis. If they are all under the x axis, or if one of them is above the x axis while the other is below the x axis, will the results be the same?

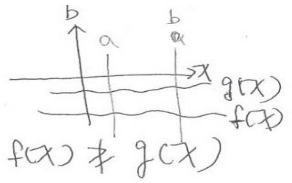
S6: They will be the same, as long as the graph of f is above that of g.

The second type of visual representation the students have developed is reflected in the graph of functions f and g, the two non-intersecting curves of which are under the x axis, but such graphs are unable to refute Statement 1. Take S9 as an example:

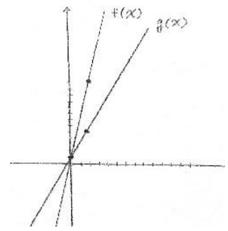
> This statement is certainly wrong. Seeing this problem, I think of area. The integral value means the area, not necessary to calculate the integral. Just as this graph (Figure 1b) I drew. From it, I found that the area surrounded by f, x = a, x = b and x axis is larger than that surrounded by g. The integral value of f is greater than that of g, but the functional value of f is less than that of g.



(1a)



(1b)



(1c)

Figure 1: Students' visual representation of Statement 1 and Statement 2

When confronting Statement 2, students employing graphical representation would immediately think of the slope of tangent under the geometric property, and then perform intuitive reasoning by replacing the size of derivatives with that of slope of tangent. Take S2 as an example:

> This statement is certainly correct, because derivative is just the slope of tangent. The derivative of f(x) is greater than that of g(x), and hence f(x)'s slope of tangent is greater than that of g(x). The simplest graph of slope is a straight line, just like this figure (Figure 1c). The slope of f(x) is greater than that of g(x), and f(x) is greater than g(x). This statement is correct.

S2 only noticed that the graph above the x axis meets the conditions of Statement 2, but failed to notice that g(x) is greater than f(x) when x is less than 0. S2 had difficulties interpreting the dynamic relationship of the basic concepts of calculus. He relied on two kinds of interconnected schema (i.e., interval and property), but he was unable to integrate them.

As suggested by Fischbein (1987), these visualizations play an important role in anticipatory solutions, as they are established on the basis of how they can be constructed and manipulated. As a result, they are conducive to converting mathematical statements into graphs.

Students' intuitive strategies can be categorized as accessibility errors (Glockner & Witteman, 2010). It is irrelevant that relevance errors take place in intellectual and narrative features. When students develop an intuitive representation of mathematical statements, the interval restrictions of narrative features that are less accessible are often neglected. This error is crucial for determining the truth value of two mathematical statements, because interval restriction is the key to determining if the two mathematical statements are wrong. Students intuitively believe that interval restrictions are irrelevant to determine the truth value of mathematical statements, so reach a wrong solution.

Similarity-based Intuitive Reasoning

The third type of intuitive reasoning used by students is Similarity-based Intuition. Students determined the truth value of a mathematical statement by replacing the relevant attributes of mathematical statements with similar attributes. Only one student used this type of intuitive reasoning when confronting Statement 1, whereas 12 students used it when confronting Statement 2. This supports the argument of Fischbein (1999) that intuitions are not absolute, they depend on the context. S7 thought Statement 1 is correct, because "I am sure this statement is correct as I met similar problems that the integral of f is bigger than or equal to that of g; by transposing and then subtracting, the integral of f will be larger than or equal to zero after subtracting that of g; so 'f minus g' is greater than or equal to zero, the proof is completed." She quoted the proving process of a theorem, "If f and g are integrable on [a, b] and if $f(x) \geq g(x), \forall x \in [a, b]$, then the definite integral of f from a to b is greater than or equal to the definite integral of g from a to b," which was proven by her teacher in class. The twelve students quoted similar attributes of various sizes of numbers, replacing the sizes of functions, to determine whether Statement 2 is correct. Take S10 as an example.

S10: When I see that the derivative of f is greater than that of g, I think of the size of numbers, e.g., 2 is greater than 1. Therefore, f(x) = 2x and g(x) = x, and 2x is greater than x.

I: Why is 2x greater than x?

S10: 2x is the double of x, so it is greater. For example, when x is equal to 1, 2 is greater than 1.

Their errors in intuitive strategies, called attribute substitution (Glockner & Witteman, 2010), occur when a more readily accessible attribute is substituted in a task for a less readily accessible attribute. For instance, similarity is a highly accessible attribute, because it is processed intuitively. S7 intuitively noticed the similarity between Statement 1 and mathematical theorems he knew. S10 noticed the similarity in the size of coefficient and function. Both of them replaced less accessible attributes with more accessible attributes. Similar to the students using property-based intuition, most of the students who noticed the similarity in the sizes of coefficient and function made relevance errors (Evans, 2010) and neglected interval restrictions.

The interviewer subsequently asked students to draw graphs of algebraic function, in order to examine whether they could overcome intuitive relevance errors after visualizing mathematical statements. Results show that eight students overcame this error after drawing function graphs. Taking S10 as an example:

S10: These are the graphs of f(x) and g(x), and f(x) is greater than g(x)...... No, that's not right. The graph below (referring to the graph below x axis) is something I didn't take notice of just now. From the graph, it can be seen

> that when x is negative, f(x) is beneath g(x), indicating that f(x) is less than g(x), so..... I am wrong. In fact, this statement is wrong. Although the derivative of f(x) (2) is greater than that of g(x) (1), it is not certain that within the interval including 0, e.g., $[-1,1]$ or $[-2,3]$, f(x) may be greater than g(x). How come I did not notice it just now? I noticed it only after drawing.

Therefore, graphical representation allowed S10 to understand the necessity of interval restrictions and overcome relevance errors. As for these students, the concreteness of visual images is an important factor for creating self-evidence and immediacy. A visual image not only organizes data at hand under a meaningful structure, but is also an important factor guiding the analytical development of a solution; visual representation serves as an important anticipatory device. The rest of the four students like S2 had difficulties in interpreting the dynamic relationship of the basic concepts of calculus. They relied on two types of interconnected schema (i.e., interval and property), but they were unable to integrate. Intuition exerts a coercive influence on the reasoning methods of individuals. An intuition subjectively generated by an individual is often a representation or interpretation that is absolute, while other representations or interpretations are excluded and unacceptable.

CONCLUDING REMARKS

The students in this study used three distinct types of intuitive reasoning to decide the truth value of mathematical statements. Each type of intuition provided students with a different starting point when approaching the tasks. With regard to logic-based intuition, we can conclude that for some students the equivalence between a statement and the converse is an intuition. With Fischbein (1982), we can remark that the *formal extrinsic type of conviction* does not seem to have any effects on the *intuitive intrinsic type of conviction*, which can remain an obstacle. When using property-based intuition, the students based their decisions on vague ideas about properties in the task and always used graphical representations. Moreover, judging the truth value of mathematical statements and generating counterexamples by visualization is mediated by the intuition of the generality of the conclusions obtained by means of it. Similarity-based intuition was used when students identified a statement that was similar enough to the given statement to suggest the truth value of the given statement, but always used symbolic representations. Students' intuitive decision on the truth value of Statement 2 supports Buchbinder and Zaslavsky's (2009) claim, is deeply rooted in the clues in the mathematical statement. From students' performance in this study, if the constructed intuitive representation accurately represents task structures, such related interpretations will have a positive effect on tasks, but if intuitive representations are distorted or deficient, they may lead to negative effects on reasoning.

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THE EFFECTS OF ESTIMATION INTERVENTIONS ON CHILDREN'S MEASUREMENT ESTIMATION PERFORMANCE

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The effects of two sets of interventions involving the same instructional approach, time and estimation strategies, but with different levels of self-checking of estimates, on fourth-grade children's immediate and retention achievements of solving problems involving length and area estimation were examined. The results displayed the treatment effects exhibited on children's retention performance rather than on the immediate achievement. Based on the interview data, the interviewees indicated that they extended their personal repertoire of benchmarks and learned to select an appropriate measure unit for improving the accuracy of their estimates.

INTRODUCTION

Using strategies for efficiently making good measurement estimates such as the guess-and-check procedure and reference point (benchmark) strategies are frequently recommended for the instruction on measurement estimation (Joram, Subrahmanyam, & Gelman, 1998). Moreover, examining the appropriateness of one's estimations is an important skill of measurement estimation (Bright, 1976), particularly, for length and area (Hildreth, 1983). Self-checking, which is an essential process of self-monitoring, may improve awareness of one's own cognitive process (Montague, 2007). Therefore, in the domain of measurement estimation, self-checking estimate activities aid children in taking control of their estimation actions, which in turn benefits the development of their measurement sense.

As for instruction on measurement estimation, Jones and Rowsey (1990), Joram, Gabriele, Bertheau, Gelman, and Subrahmanyam (2005) and Jones, Taylor, and Broadwell (2009) all examined the immediate effects estimation strategy instruction on students' measurement estimation abilities. Although Jones and Rowsey provided a brief discussion of seventh-grade students' retention of the estimation skill applications, what the delayed effects of instructional interventions involving estimation strategies are on elementary school children's retention of measurement estimation skills remains unclear. In the meantime, self-checking estimate activities which are recommended for nurturing children's estimation ability (Bright, 1976; Hildreth, 1983), have not been highlighted in the previous studies mentioned above. It is therefore worth exploring the role that self-checking plays in children's measurement estimation.

The use of interviews to collect learners' verbal explanations of their uses of strategies learned from interventions is a suitable method for understanding children's mathematical conceptions and problem-solving strategies (Huang & Witz, 2011). Thus, to understand what skills children obtained from the interventions for solving measurement estimation problems, interviews were included in this study.

The present study aims at examining the effects of the interventions that contained concepts of measurement estimation and strategies, but with different levels of self-checking of estimates, on promoting fourth-grade children's performance of measurement estimation, particularly, in the domain of length and area. The research questions included in the study are presented as follows:.

1. What are the effects of the interventions on children's immediate achievements of measurement estimation?
2. What are the effects of the interventions on children's retention achievements of measurement estimation?
3. What are estimation skills that the children obtained from the interventions?

THEORETICAL FRAMEWORK

Mathematical thinking involved in measurement estimation

In mathematics, to make a measurement estimate means to determine a quantitative value of an object without using a measuring tool such as length or area (Bright, 1976). To mentally measure a to-be-measured (TBE) object, the process of estimation includes using a known unit of measure as a mental reference unit, repeatedly comparing the unit with the object mentally, and then computing a quantitative answer (Jones, Taylor, & Broadwell, 2009).

For making length and area estimations, proportional reasoning (Jones et al., 2009) and visual-spatial thinking (Joram et al., 1998) are demanded. Both of those types of thinking are complex cognitive skills, which may develop with increase in age and learning experience (Tourniaire & Pulos, 1985).

Strategies and self-checking of estimates in measurement estimation

Guess-and-check and the use of body parts as reference points are commonly recommended in length measurement activities in Taiwan (Huang, 2015). The guess-and-check procedure includes guessing (or thinking of) the size of a unit for estimating and judging the number of units needed to replicate the size of the TBE object, and then checking the answer by actually measuring the object. In contrast, using reference points for performing estimation involves imaging and comparing an object for which the measurement is known with the TBE object (Joram et al., 2005).

In addition to the uses of estimation strategies, Bright (1976) suggested that self-checking after estimating provides "an experiential background from which errors in measuring can be explained and accurate measuring skills can be isolated, studied, and improved" (p. 94). The process of self-checking, which is highly related to awareness of one's own cognitive processes, may help the development of one's feelings of knowing and retrospective judgment of performance (Montague, 2007). As Bright

(1976) suggested, self-checking of one's estimates assists in discriminating the measuring action from the abstract concept of measurement.

Previous studies on the interventions involving various estimation strategies

In the field of length, mass, and capacity measurements, Jones and Rowsey (1990) compared a treatment group that received measurement instruction involving the estimation strategy (guess-and-check procedure) with a control group that focused only on direct measurements without any estimation strategies. Jones and Rowsey found no differences in the immediate achievements between the two groups, but did find treatment effects exhibited in the students' retention performance of metric applications, which was assessed five weeks after the post-test. This implies that the development of estimation competence takes some time.

Moreover, Joram et al. (2005) indicated that a group which received treatment involving the use of the benchmark strategy performed better on length estimation than another group which received treatment involving the guess-and-check procedure without using the benchmark strategy. Jones et al. (2009) reported that students gained improvement in estimating linear size and scale after receiving the treatment involving the use of body rulers.

In sum, providing instruction that includes either the guess-and-check procedure or the use of benchmark strategy for measurement estimation may improve children's immediate achievements. However, what the delayed effects of treatments involving estimation strategies on children's retention of estimation skills are uncertain.

METHODOLOGY

A quasi-experimental design was used to examine the effects of two interventions involving the same instructional approach (guided instruction approach), teaching time, and estimation strategies, but different levels of self-checking of estimates, on children's immediate and retention achievements. Moreover, one-on-one interviews were conducted to understand what estimation skills the children learned from the interventions provided.

Participants

In the study, three fourth-grade classes (N = 88) were recruited from a public elementary school in Taipei, Taiwan. One of the three participating classes, which was the instructor's homeroom class, served as the control group (n = 20). The two sets of interventions were randomly assigned to the remaining two classes. One of these two classes (n = 35) received the experimental curriculum stressing estimation with self-checking of estimates post estimation, while the other class (n = 33) was provided with the experimental curriculum involving estimation with a low level of self-checking of estimates post estimation. All of the participants had learned length and area measurements with some exposure to length estimation mainly provided by their mathematics textbooks in previous lessons.

Instruments

Two sets of experimental curricula and three assessments (i.e., the pre-test, post-test, and retention test) were administered in the study. The series of measurement estimation tasks developed for the experimental curricula and the questions designed for the assessments were based on the theoretical framework (e.g., Bright, 1976; Jones et al., 2009; Joram et al., 1998) and the previous studies on measurement estimation (Huang, 2015, in press). One set of the experimental curricula highlighted the uses of estimation strategies with self-checking of estimates post estimation (abbreviated as the ESC curriculum hereafter), whereas the other involved the use of estimation strategies that are the same as those in the ESC but with fewer occasions for the self-checking of estimates (abbreviated as the EST curriculum hereafter) than those provided in the ESC curriculum. Each intervention was carried out for 5 class-periods. Most of the TBE objects described in the tasks of the curricula or assessment questions were visually presented to the participants using real objects or figures.

During the experimental period, the control group studied a textbook unit which involved computations with three-to-four-digit numbers and computational estimation excluding measurement estimation, based on the regular schedule.

The sets of experimental curricula consisted of four main components as follows. (A1) The meaning of estimation and how estimation plays a role in measurement and estimating length and area measures. (A2) The use of metric-system units (e.g., cm, m, and cm^2) and the language of estimation such as “about,” “close to,” and “between.” (A3) The uses of the estimation strategies for solving estimation problems: (a) the guess-and-check procedure and (b) the benchmark strategy. Moreover, Jones and Taylor (2009) suggested that the physical act of moving around in a field may help students learn about size and scale. In the study, in addition to using a known unit to mentally perform unit iterations, recalling the known size of a unit and performing unit iteration through gestures or physical movements were allowed in each intervention. (A4) Self-checking of estimates. To recognize the difference between estimates and the actual measurement of a TBE object, self-checking of estimates has been suggested as a feasible approach for checking the errors in measuring (Bright, 1976). The specific features of each intervention were indicated are indicated as follows.

The ESC intervention. The treatment implemented 16 tasks underlying components A1 to A4 with an emphasis on A4. Each task was provided with a table that required recordings of the estimate and the actual measurement of the TBE object and the difference between the two measures. Children who received the ESC curriculum were requested to fill the estimation results in the tables and check the reasonableness of the estimate during post-estimation discussion conducted by the instructor. In the study, the tables served to assist self-checking of the estimates.

The EST intervention. The treatment implemented 21 tasks underlying components A1 to A4 but with a minor emphasis on A4. About one-third of the estimation tasks were provided with tables for self-checking of estimates as given in the ESC curriculum.

The process of discussion for checking the estimates post-estimation administered in the EST group was similar to that in the ESC group.

In the study, the pre-test, the post-test, and the retention test were equivalent assessments. The pre-test was used to examine the children's ability to measurement estimation prior to the interventions, whereas the post-test, which was undertaken within a week after completing the interventions, was used to assess their immediate achievements. To measure the children's retention of the applications of their estimation skills, the retention test was conducted about five weeks after the post-test.

Each test consisted of 13 questions. There were four or six questions that required estimating large TBE objects. For example, for the length questions, a TBE object with length exceeding 100 cm was defined as "large." For the area questions, a TBE object with area larger than 1,000 cm^2 was regarded as "large." The number of items that demanded a large estimated quantity included in the pre-test, post-test, and retention test was six, four, and six, respectively.

In order to understand the children's learning gains from the interventions, one-on-one interviews were conducted after the interventions. The interviews were audio taped and transcribed for analysis. The present paper focuses on the block of questions regarding learning gains, that is, "Did you change your initial methods used for estimating length to make your estimates more reasonable after the instruction? Why? If you did not change your initial methods, why not?" The analysis of the interview data was based on a pair of interviewees recruited from each experimental group based on the scores of the post-test. Each pair included one high-achiever and one low-achiever.

Scoring

In the study, a "reasonable" estimate was defined as being within ± 10% of the actual value, as described by Huang (in press), and was scored 2 points. An "acceptable" estimate was defined as being between +10% and +25% or -10% and -25% of the actual value and was scored 1 point. If an estimate was greater than +25% or lower than -25% of the actual value, then a score of "0" was allocated.

The level of difficulty of estimating a large quantity is greater than that of estimating a small quantity (Huang, in press). Hence, a weighted method was used for scoring the questions with large TBE objects in each test. That is, when scoring the questions requiring a large estimated quantity, weighted scores were given to a reasonable estimate (4-points) and an acceptable estimate (2-points), respectively. The maximum total scores of the pre-test, post-test, and retention test were 42 points, 38 points, and 42 points, respectively.

RESULTS

The comparisons of the immediate achievements and retention performance

The means of the total scores and standard deviations of the pre-test, post-test, retention test, and the adjusted means of the post-test and retention test by group are displayed in Table 1. As can be seen in Table 1, the total scores of the pre-test obtained by the three groups from high to low are the control group, the EST group, and the ESC group. To compare the effectiveness of the two interventions (ESC vs. EST) on the children's immediate achievements and retention performance of measurement estimation, ANCOVAs with the pre-test score as the covariate were executed.

Table 1. The mean scores of the pre-test, post-test, and retention test by group

	n	Pre-test	Post-test		Retention test	
		M (*SD*)	*M* (*SD*)	Adjusted *M*	*M* (*SD*)	Adjusted *M*
ESC group	35	16.46 (6.25)	17.80 (5.59)	18.35	26.34 (5.27)	26.79
EST group	33	17.61 (5.92)	18.36 (4.67)	18.43	26.15 (4.91)	26.21
Control group	20	19.25 (5.87)	18.95 (6.55)	18.34	21.20 (6.75)	20.69

For the immediate achievements, the results of the ANCOVAs with the pre-test score as covariate showed no significant difference among the three groups, F (2, 84) = .001, p = .99. The results indicated that the two experimental groups did not obtain higher immediate achievements on the post-test compared to the control group. Moreover, the post-test scores of the ESC group were close to those of the EST group.

For the retention performance, the results of the ANCOVAs with the pre-test score as covariate showed a significant main effect of treatment, F (2, 84) = 9.65, $p < .01$, partial $\eta^2 = .19$. The results of pair comparisons among the three groups revealed that both the ESC and the EST groups outperformed the control group. However, no differences were found between the ESC group and EST group.

The interviewees' learning gains obtained from the interventions

Based on the interview data, the four interviewees expressed that they extended their personal repertoire of benchmarks and learned to select an appropriate measure unit for improving the accuracy of their estimates. For example, the low-achieving interviewee from the EST group expressed that at first she just used the length of her outstretched thumb and index finger as her only one reference point for estimating, but then learned to use multiple body parts as measures depending on the size of the TBE objects, for example, the width of a finger, an outstretched arm and two open arms. Similarly, the high-achieving interviewee from the EST group expressed that "I did not know how to estimate but used a ruler at the beginning of the classes. Afterwards, I learned the use of body parts." The body parts that she indicated included the length of a little finger and the length of her outstretched thumb and index finger, body length, foot-steps, and open arms.

Moreover, the four interviewees did in fact change their initial estimation methods after the interventions. For example, the low-achieving interviewee from the ESC group indicated the changes of using an eraser as a measure unit initially to using the length of his outstretched thumb and index finger with unit iteration afterwards. The high-achieving interviewee from the ESC group indicated that,

> "Prior to the classes and in the first session, I estimated visually because I did not know what else can be used… I then learned to use known items as references. For example, this item, I knew it though I estimated it visually before…I used my eyes all the time but it is possible to produce a big margin of error… Now I frequently use the ways I know. For example, I used the length of my palm and the width of a finger to measure when measuring a short rope before. Now I attempt to watch the objects around me and (select one item to) compare (the object). "

In sum, the common consideration of the interviewees for changing their original methods was to improve the accuracy of the estimate made by the individual. Furthermore, three of the four interviewees addressed changing methods to improve their efficiency (i.e. obtaining a reasonable estimate fast).

DISCUSSION AND IMPLICATION

The findings of the study exhibited that the two experimental groups which received the interventions involving estimation strategies for length and area estimations with different levels of self-checking did not outperform the control group, which received the textbook unit involving computational estimation, on the immediate achievements. In contrast, the two experimental groups obtained higher scores than the control group on the retention test. These findings imply that both the interventions showed delayed effects rather than immediate effects on improving the children's performance of measurement estimation. The results seem to echo the findings of Jones and Rowsey (1990).

Kwon, Lawson, Chung, and Kim (2000) suggested that developing complex cognitive skills such as reasoning requires prefrontal maturity and instructional activity that provides sufficient experiences of physical manipulation with verbal interaction. Children may need a period of time for elaborating and transforming learning experience to improve their reasoning skills (Tourniare & Pulos, 1985). Thus, in this study, the two sets of interventions involving physical manipulations and verbal discussion for solving estimation problems showed delayed effects on the children's retention performance rather than immediate effects on their performance of the post-test.

It was found that there were no differences in the immediate and retention achievements of the ESC and the EST groups which received the same estimation strategies (guess-and-check and benchmark strategies) but with different levels of self-checking. In this study, the same instruction time was given to each group, but the EST group engaged in a lower level of self-checking activities than the ESC group did. In contrast, the EST group undertook a little more estimation practice through solving the estimation tasks (21 tasks) than the ESC group (16 tasks) did. Although self-checking estimate activities may help enhance children's estimation skills, such skills may also be taught and retained through more practice (Joram et al., 1998). Thus, the ESC group and EST group performed equivalently on both tests after instruction. Such gains can also be supported by the interview data.

Interestingly, the interviewees from the two experimental groups constructed new reference points from the given interventions, they tended to address the use of body parts as measure units and performed the unit repeated strategy through gestures rather than making mental estimations. To help children make mental estimations, the factors that may lead to their preference for the unit repeated strategy with physical movement may need further studies.

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