# Four-component decomposition of sense making in algebra

# Four-component decomposition of sense making in algebra

Alik Palatnik[1] and Boris Koichu[2]

[1] Shaanan Academic Religious Teachers' College, Haifa, Israel; umapalatnik@gmail.com

[2] Weizmann Institute of Science; boris.koichu@weizmann.ac.il

*This article presents a case in which a pair of middle-school students attempts to make sense of a previously obtained by them position formula for a particular numerical sequence. The exploration of the sequence occurred in the context of two-month-long student research project. The data were collected from the students' drafts, audiotaped meetings of the students with the teacher and a follow-up interview. The data analysis was aimed at identification and characterization of events and algebraic activities in which the students were engaged while making sense of the formula. We found that the students' conviction, by the end of the project, that the formula "makes sense" emerged when they justified the formula, checked its generality, discovered a geometry mechanism behind it, and found that it came to cohere with additional formulas. The findings are summarized as a suggestion for a four-component decomposition of algebraic sense making.*

*Keywords: Algebraic sense-making, problem-solving, project-based learning, integer sequences.*

## Introduction

Sense making has long been a focal concern of the mathematics education research community (e.g., Kieran, 2007; NCTM, 2009). NCTM (2009) recognised sense making as a means to know mathematics as well as an important outcome of mathematics instruction. To review, NCTM (2009) refers to *sense making* in mathematics "as developing understanding of a situation, context, or concept by connecting it with existing knowledge" (p. 4). Nevertheless, NCTM (2009), as well as many additional mathematics education publications, is rather inexplicit as to what sense making comprises of and how it occurs. Moreover, it has been broadly acknowledged (e.g., Schoenfeld, 2013) that empirically-based knowledge about the processes involved in sense making, as well as knowledge about the processes involved in learning through mathematical problem solving, is insufficient.

The case presented in this article occurred with two 9$^{th}$ graders, Ron and Arik (pseudonyms) who participated in the Open-Ended Mathematical Problems project, which was conducted by the authors (an abridged version of Palatnik and Koichu, 2017). The initial part of Ron and Arik's project lasted for three weeks and resulted in an insight solution to the problem of finding a position formula for a particular sequence. This part is analysed elsewhere (Palatnik & Koichu, 2015). The insight gained was celebrated as an important highlight of the project. The students told us, however, that they found the formula "by chance" and that it did not make sense for them. As a result, making sense of the obtained formula became an explicitly chosen goal and the main theme of the second part of the students' project. This part had lasted for four weeks and ended when the students succeeded, in quite an idiosyncratic way, to make sense of the formula.

The goal of our study was to discern the activities and processes involved in the sense making effort. Specifically, we pursued the following research questions:

In which events and algebraic activities were the students engaged while attempting to make sense of a formula?

What were some of the processes involved in the students' explicitly expressed conviction, by the end of the exploration, that the formula "makes sense"?

## Theoretical background

In empirical studies, the notion of sense making frequently denotes ways by which learners of mathematics act upon a particular entity in the context of particular mathematical activity. The expression "to make sense of…" is attributed in different studies to such entities as proofs, instructional devices, concepts, solution methods and problem situations (e.g., Smith, 2006; Rojano, Filloy, & Puig, 2014).

The idea of algebra as an activity was elaborated by Kieran (1996, 2007). Kieran identifies three types of activities in school algebra: *generational*, *transformational*, and *global/meta-level* activities, and argues that each type has special affordances to meaning construction. The *generational activity* involves the forming of the objects of algebra (e.g., algebraic expressions or formulas) including objects expressing generality arising from geometric patterns or numerical sequences. The *transformational activity* includes various types of algebraic manipulations. Transformational activity can involve meaning construction for properties and axioms on which the manipulations rely. A related point is highlighted by Hoch and Dreyfus (2006), who proposed the notion of *structure sense*, which is related to algebraic manipulations and aspects of *symbol sense* (Arcavi, 2005) in relation to *friendliness* with symbols as tools, an ability to switch between attachment and detachment of meaning, and an examination of the meaning of symbols. Finally, Kieran (2007) argues that meaning construction is associated with *global/meta-level mathematical activities* (e.g., problem solving, working with generalizable patterns) in a sense that "these activities provide the context, sense of purpose, and motivation for engaging in the previously described generational and transformation activities" (p. 714). It is essential for the forthcoming analysis that when the learners are engaged in a global/meta-level activity, they can carry it out in a variety of ways, and the decision to use the algebraic apparatus arises as learners' choice.

Treatment of sense making as an inseparable part of mathematical thinking makes the MGA model of creating mathematical abstractions (Mason, 1989) particularly important for our study. The main operational categories of the model are Manipulating, Getting-a-sense-of, Articulating (hence MGA). The MGA model elaborates on the processes of creating abstraction as a helix, in which each cycle includes its own, local, sense-making act. Briefly, the model presumes that **m**anipulating familiar mathematical objects (M) leads to the formation of a sense of **g**enerality or regularity based on properties of these objects (G), and then to the **a**rticulation of that general property or regularity (A), which in turn forms new objects for further manipulations. Mason (1989) suggested that the driving force behind the process of creating abstractions is the gap between expected and actual results of manipulations.

To summarize, in our study we adapt NCTM's (2009) perspective on sense making, and elaborate on it in an algebraic context. Our theoretical framework is built upon the idea of algebra as an activity (Kieran, 1996, 2007) and on analytical apparatus of Mason's (1989) model of mathematical thinking known as Manipulating – Getting-a-sense-of –Articulating (MGA).

# Method

## Learning environment, participants and the mathematical context

The Open-ended Mathematical Problems project, in the context of which the case of Ron and Arik took place, is being conducted, since 2010, in 9th grade classes for mathematically promising students. The learning goal of the project is to create for students a long-term opportunity for developing algebraic reasoning in the context of numerical sequences. It is of note that 9th graders in Israel, as a rule, do not possess any systematic knowledge of sequences; this topic is taught in the 10th grade.

The project is designed in accordance with the principles of the Project-Based Learning (PBL) instructional approach (e.g., Blumenfeld et al.,1991). Specifically, the organizational framework of the project is as follows. At the beginning of a yearly cycle of the project, a class is exposed to 8-10 challenging problems. The students choose one problem and work on it in teams of two or three. They work on the problem at home and during their enrichment classes. Weekly 20-minute meetings of each team with the instructor (the first author) take place during the enrichment classes. When the initial problem is solved, students are encouraged to pose and solve follow-up problems. At the end of the project, all teams present their results to their peers. Then 4-6 teams, chosen by their classmates, present their work at a workshop at the Technion – Israel Institute of Technology, attended by academic audience (for more details see Palatnik, 2016).

Ron and Arik chose to pursue the Pizza Problem (Figure 1) which is a variation of the problem of partitioning the plane by n lines (e.g., Pólya, 1954).

> Every straight cut divides a pizza into two separate pieces. What is the largest number of pieces that can be obtained by $n$ straight cuts?
>
> A. Solve for $n = 1, 2, 3, 4, 5, 6$.
>
> B. Find a recursive formula for the $n^{th}$ term of the sequence.
>
> C. Find a position formula for the $n^{th}$ term of the sequence.

**Figure 1: The Pizza Problem**

Using Kieran's (2007) terminology, we expected the PBL environment and the Pizza Problem in particular to afford students to be engaged with generational and transformational activities in the context of a global/meta-level activity. In this way, the students were provided with opportunities for developing algebraic sense-making and we – with an opportunity to study their sense-making effort.

## Data sources and analysis

We audiotaped and transcribed protocols of the weekly meetings with Ron and Arik (eight 20-minute meetings), collected written reports and authentic drafts that the students prepared for and updated during the meetings (more than 40 pages) and interviewed the students by the end of the project. These data were used to create a description of the students' exploration and for dividing it into events.

In accordance with the presented above methodological principles for exploring sense making, we discerned the activities the students chose to be engaged in: proving, generalizing, pattern-seeking, and question-generating. We also applied the MGA model to trace mathematical objects manipulated by the students in a sequence of activities potentially contributing to sense-making.

## Findings: Ron and Arik make sense of the obtained formula

We present here four main events that occurred during students' sense-making pursuit.

### Event 1: Choosing new goals

The following conversation took place just after the students presented their solution of the Pizza Problem to the instructor:

Instructor: Now you have a lot of work to do, and this is great. First of all, you see that the formula works. Now we have to think why it works, and try proving that it works.

Ron (to Arik): Write it down. "Why it works, and prove that it works" (laughs), it is interesting!

Ron accepted instructor's suggestion. In his words: "When we have a formula, but don't know its meaning, it is not interesting. If we knew how the formula is constructed, we would know it 100%. We got it by chance. So we do not know what it means." In addition, both students proposed to explore a more general problem, that of plane partitioning (see Figure 2).

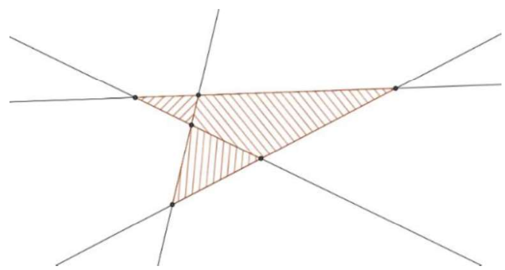
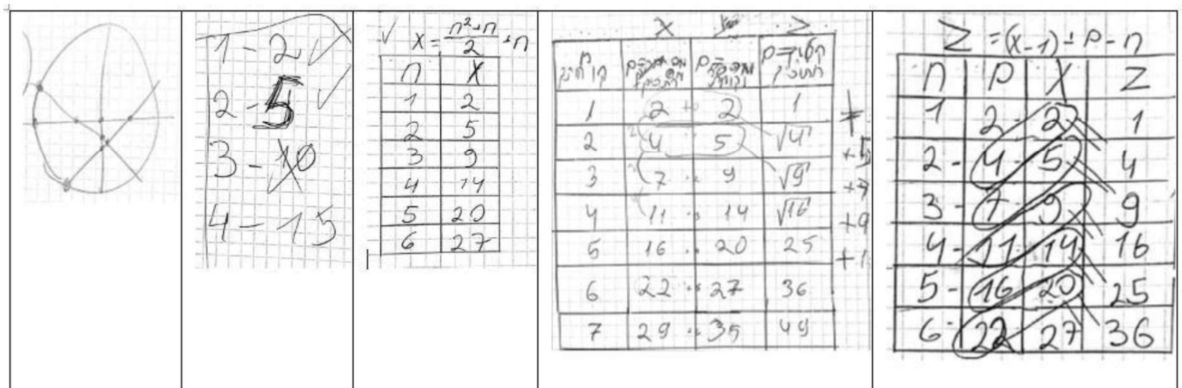


**Figure 2: Division of the plane: there are three closed (hatched) and eight open pieces**

The students also suggested additional objects to explore: the points of intersection of the cutting lines with and within a circle representing a pizza and the number of segments on the cutting lines.

### Event 2: Simultaneous exploration of several sequences and first manipulation with a formula

Having chosen the above goals, the students started making sketches and counting: segments within the circle, closed and open parts of the plane and points of intersection of the cutting lines, for different numbers of lines (see Figure 3a-c). As a by-product, the students noticed that the sum of the first $n$ odd numbers also equals $n^2$. They also began exploring the connections between different sequences (see Figure 3d-3e). In particular, Ron noticed that the differences between the corresponding terms of the sequences form a sequence 0, 1, 2, 3... (see columns X,Y at Figure 3d).



| 3a | 3b | 3c | 3d | 3e |
|---|---|---|---|---|

**Figure 3: The strategy employed in Events 1 and 2**

To obtain an explicit formula for the sequence 2, 5, 9, 14 … (the numbers of intersections of the cutting lines with and within the circle), Ron *adjusted* the formula $P_n = \frac{n^2+n}{2} + 1$ into the formula $X = \frac{n^2+n}{2} + n$ (Figure 3c) in the following way: "I thought it would be like the previous formula, but it did not fit. So I got rid of 1 and added *n* [to the right side of the formula], and it was right."

**Event 3: Producing an explanation of why the target formulas worked**

The wish to understand why the formula returns the maximum number of pieces was a repeated theme in weekly meetings with the instructor. The students eventually answered this query in the following way. After exploring of new drawings Ron and Arik realized that the maximal number of pieces is obtained when a new cutting line crosses *all* the previous lines in new points. As a result, the students concluded that a new cutting line added *n* new intersection points to the existing configuration of lines. For the students, it was an explanation of why the formula $X = \frac{n^2+n}{2} + n$ returned the *maximum* number of the intersection points. They further asserted that this idea also explained, for them, why the target formula $P_n = \frac{n^2+n}{2} + 1$ returns the *maximum* number of pieces.

**Event 4: "Proving" the target formula**

As mentioned, the need to prove the correctness of the formula for the Pizza Problem was an additional driving force for the students. First, Ron suggested: "We thought of a way to prove it [the position formula]…[in order to do so, we wanted] to connect all the formulas we had, every table we've made… may be it will give us the formula, then we will know that it is a true formula indeed. Then we'd have a proof". Ron and Arik built upon the following inference: for any number of cuts, the sum of the number of open and closed pieces (see Figure 2) equals the overall number of pieces into which a plane is divided. They explored the sequences for open and closed pieces. The number of open pieces for *n* cuts, 2*n,* was easy for them to find and explain: adding a new cutting line adds exactly two open pieces to the drawing. For the closed pieces the students empirically (i.e., by counting on the drawings) obtained a sequence 0, 0, 1, 3, 6 for 1, 2, 3, 4 and 5 cuts, respectively. They perceived it as "quite close" to the target sequence (2, 4, 7, 11, 16…) and began manipulating the target formula ( $P_n = \frac{n^2+n}{2} + 1$ ) in a way similar to *adjustment* in Event 2. Eventually Ron and Arik obtained the correct expression $\frac{(n-2)^2+n}{2} - 1$. The last piece of the puzzle came when Ron and Arik and their classmate with whom they consulted devised and realized the following plan. Since the formula $P_n = \frac{n^2+n}{2} + 1$ represents the total numbers of pieces and since Ron and Arik have obtained the formulas for the numbers of closed and open pieces, the three formulas should match. After several unsuccessful attempts, Ron and Arik implemented this idea and algebraically connected the three formulas. In their final presentation, they showed a slide with the following transformations: $\frac{(n-2)^2+n}{2} - 1 + 2n = \frac{n^2-4n+4+n-2+4n}{2} = \frac{n^2+n+2}{2}$; $\frac{n \cdot (n+1)}{2} + 1 = \frac{n^2+n+2}{2}$. This and validation of all three formulas by means of Excel tables were presented as "the proof" of the target formula, and the formula itself was treated as "making sense" by the students.

## Discussion

The four-weeks-long exploration of two 9th grade students working on a particular project has been presented. The answer to the first research question (about events and algebraic activities in which the students were engaged while attempting to make sense of the previously obtained position formula) straightforwardly follows from the above exposition. Briefly, the students were engaged in generational and transformational activities in the context of the global/meta-level activities of explaining to themselves why the formula worked and of proving the formula. It is of note that Ron and Arik's persistence to make sense of their formula is unusual. We suggest two circumstances contributing to the emergence of the students' self-imposed sense-making goal. First, the students' activities were organized and driven by their interest to a particular mathematical phenomenon and not merely to generation of some patterns (cf. Hewitt, 1992, for train spotters metaphor). Second circumstance is the organizational setting of the project, which was in accordance with project-based learning instructional approach (Blumenfeld et al., 1991). In such an environment the students had a chance to get used to the long-term, open-ended explorations, to the high level of expectations and to having room and time to spend with a problem.

Our second research question concerned the processes involved in student sense making. To answer the query "why the formula works" the students examined *the geometric mechanism* behind the formula. In the course of generational activity the students experimented with concrete drawings (i.e., drawings with 4-6 cutting lines), which apparently served as a visual tool to reveal a generic process that occurs when a line is added to a system of *n* existing lines. Accordingly, the multi-stage process of abstracting, at each stage of which an MGA cycle occurred, seems to be the central process underlying the why-part of the students' sense-making effort.

The query "how to prove the formula" turned to be the thorniest part of the project. The students addressed this query when they succeeded to show how the target formula came to *cohere* with two geometrically related formulas. These formulas were obtained by means of exploration of the connections between the sequences chosen by students. The connections were found in the process that featured counting on the drawings, pattern-sniffing in the tables and manipulating the previously obtained formulas by adjusting them. Eventually, the target formula was inserted in a cloud of related formulas, which did not exist when the students began the sense-making pursuit.

Thus, the process of generating a cloud of formulas and checking it for coherence seems to be an important process in the proving part of the students' sense-making effort (cf. Rohano, Filloy & Puig, 2014, for sense making by connection of a new mathematical text to a system of texts). It is of note that the coherence was achieved not only among various objects, but also by means of a coherent exploration strategy.

As argued, Ron and Arik constructed meaning of the target formula in a sense-making process consisting of sequence of generational and transformational algebraic activities in the overarching context of global, meta-level activity, long-term problem solving. In this sense-making process, the students: (1) formulated and justified claims; (2) made generalizations, (3) found the mechanisms behind the algebraic objects (i.e., answered why-questions); and (4) established coherence among the explored objects. We now take the liberty of formulating this summary as a proposal for a four-component decomposition of sense making (see Figure 4).

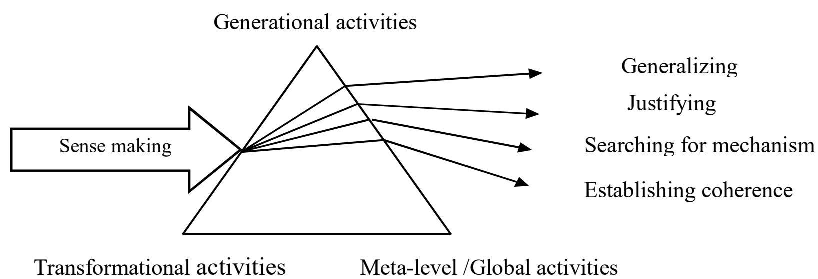



**Figure 4: Four aspects of an algebraic sense making through algebraic activities**

The aspects of *generalizing*, *justifying* and *search for mechanism* in sense making are in line with the main attributes of symbol sense (Arcavi, 2005) as well as findings about the role of generalizing and justifying in meaning construction (e.g., Lannin, 2005; Radford, 2010). However, *establishing coherence* has not yet been considered as part of sense making.

The four-component decomposition elaborates NCTM's (2009) definition of sense making in the following way. First, it presents sense making as a conjunction of processes. Second, it highlights the potential of algebraic activities to provide students with means to make sense of algebraic objects.

## References

Arcavi, A. (2005). Developing and using symbol sense in mathematics. *For the Learning of Mathematics*, *25*(2), 42−47.

Blumenfeld, P., Soloway, E., Marx, R., Krajcik, J., Guzial, M., & Palinscar, A. (1991). Motivating project-based learning: Sustaining the doing, supporting the learning. *Educational Psychologist*, *26*, 369−398.

Hewitt, D. (1992). Train spotters' paradise. *Mathematics Teaching*, *140*, 6–8.

Hoch, M., & Dreyfus, T. (2006). Structure sense versus manipulation skills: an unexpected result. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th*

*Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 305−312). Prague, Czech Republic: PME.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *Eighth International Congress on Mathematical Education: Selected lectures* (pp. 271–290). Seville, Spain: S.A.E.M. Thales.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Charlotte, NC: Information Age.

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning, 7*(3), 231−258.

Mason, J. (1989). Mathematical abstraction as the result of a delicate shift of attention. *For the Learning of Mathematics*, *9*(2), 2−8.

Meira, L. (1998). Making sense of instructional devices: The emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education*, *29*, 121−142.

NCTM (National Council of Teachers of Mathematics) (2009). *Executive summary: Focus in high school mathematics: Reasoning and sense-making*. Reston, VA: The Author.

Palatnik, A., & Koichu, B. (2015). Exploring insight: focus on shifts of attention. *For the Learning of Mathematics*, *35*(2), 9−14.

Palatnik, A., & Koichu, B. (2017). Sense making in the context of algebraic activities. *Educational Studies of Mathematics*, doi:10.1007/s10649-016-9744-1.

Palatnik, A. (2016) *Learning through long-term mathematical research projects*. (Unpublished doctoral dissertation). Technion – Israel Institute of Technology, Haifa, Israel.

Pólya, G., (1954). *Induction and analogy in mathematics*. Princeton, NJ: Princeton University Press.

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA - Pensamiento Numérico Avanzado*, *4*(2), 37−62.

Rojano, T., Filloy, E., & Puig, L. (2014). Intertextuality and sense production in the learning of algebraic methods. *Educational Studies in Mathematics*, *87*, 389−407.

Schoenfeld, A. H. (2013). Reflections on problem solving theory and practice. *The Mathematics Enthusiast*, 10 (1-2), 9−34.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *The Journal of Mathematical Behavior*, *25*, 73−90.